MINIMAL CLONES

PhD Dissertation

Waldhauser Tamás

Advisors:
Dr. Csákány Béla
Dr. Szendrei Ágnes

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Contents

Chapter 1

Introduction

We review the rudiments of minimal clones in this chapter. In Section 1.1 we discuss clones of functions, the five types of minimal clones and the basic tools for proving the (non)minimality of a clone. Section 1.2 explains the connections between varieties and clones, and introduces the technique of absorption identities. In Section 1.3 we give some examples of minimal clones and recall some of their properties, and in the Section 1.4 we mention some characterization theorems about minimal clones.

1.1 Concrete clones

A set $\mathcal{C}$ of finitary operations on a set A is a *(concrete) clone*, if it is closed under composition of functions and contains all projections. The *composition* of an n-ary function f by functions $g_1, \ldots, g_n$ of arity k is the k-ary function $f(g_1, \ldots, g_n)$ defined by

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_k) = f(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k)),$$

and the i-th n-ary *projection* is the function

$$e_i^{(n)} : A^n \to A,\ (x_1, \ldots, x_n) \mapsto x_i \quad (i = 1, \ldots, n).$$

If $\mathbb{A} = (A; F)$ is an algebra, then the set of its term functions, denoted by $\operatorname{Clo} \mathbb{A}$, is a clone on A, called the *clone of the algebra* $\mathbb{A}$. This is the smallest clone containing F, therefore we say that F *generates* $\operatorname{Clo} \mathbb{A}$, and we write $[F] = \operatorname{Clo} \mathbb{A}$. Clearly, every clone arises as the clone of an algebra: we just need to pick a generating set for the clone, and let these be the basic operations of the algebra. If $g \in [F]$, then there is a term t such that g is the term function of the algebra $(A; F)$ corresponding to t. In this case we will simply say that t is a term of type F, and we write $g = t^F$. With this notation we have

$$g \in [F] \iff (\exists t) : g = t^F.$$

It is a basic problem to decide whether $g \in [F]$ holds for given g and F or not. The affirmative answer can be proved by finding the appropriate term t for which $g = t^F$. Relations provide a tool for establishing the negative answer. We say that a function f (a set F of functions) *preserves* the k-ary relation ρ, if ρ is a subuniverse of $\mathbb{A}^k$ for $\mathbb{A} = (A; f)$ ($\mathbb{A} = (A; F)$). The following fact is the key for proving $g \notin [F]$.

$$F \text{ preserves } \rho \ \text{ and } \ g \in [F] \implies g \text{ preserves } \rho \tag{1.1}$$

Thus relations are obstacles for producing functions from other functions by compositions, moreover, if A is finite then they form a complete set of obstacles: if $g \notin [F]$, then there exists a relation ρ such that F preserves ρ, but g does not. This is a consequence of the so-called Pol-Inv Galois connection between functions and relations [BKKR, Ge]. We will not present the details here, as we need only (1.1); in fact, we will always find a suitable unary relation or an equivalence relation (i.e. a subuniverse or a congruence of the corresponding algebra).

All clones on a given set A form a lattice with respect to inclusion; the smallest element of this lattice is the *trivial clone*, the clone of all projections on A, while the greatest element is the clone of all finitary operations on A. These clones will be denoted by $\mathcal{I}_A$ and $\mathcal{O}_A$ respectively; the subscripts will be sometimes omitted, if the base set is clear from the context. The elements of the trivial clone (the projections) will be referred to as *trivial functions*, and we say that $\mathbb{A}$ is a *trivial algebra* if its basic operations are all trivial, i.e. if $\operatorname{Clo}\mathbb{A} = \mathcal{I}$. Note that this is different from the usual notion of triviality: one-element algebras are trivial in this sense, too, but not only those. For example, a groupoid is trivial iff it is a left or right zero semigroup, regardless of its size.

Minimal clones are the atoms of the clone lattice, i.e. a clone is minimal if its only proper subclone is the trivial clone. On finite sets there are finitely many minimal clones, and every clone contains a minimal one (cf. [PK, Qu2, SzÁ]). Clearly, a nontrivial clone is minimal iff it is generated by any of its nontrivial elements:

$$\mathcal{I} \neq \mathcal{C} \text{ is minimal } \iff (\forall g \in \mathcal{C} \setminus \mathcal{I}) : \mathcal{C} = [g]\,. \tag{1.2}$$

Therefore all minimal clones are one-generated, thus they arise as clones of algebras with a single basic operation. We usually define a minimal clone by a generating function, so let us reformulate (1.2) accordingly:

$$\mathcal{I} \neq [f] \text{ is minimal } \iff (\forall g \in [f] \setminus \mathcal{I}) : f \in [g]\,. \tag{1.3}$$

Taking into account that the clone generated by a function can be described in terms of terms [!] we can express (1.2) yet another way:

$$\mathcal{I} \neq [f] \text{ is minimal} \iff (\forall t_1) : \Big(g = t_1^f \notin \mathcal{I} \implies (\exists t_2) : f = t_2^g \Big), \tag{1.4}$$

where t_1 and t_2 are terms of type f and g, respectively. Note that the validity of this formula can be decided just by taking a look at the identities satisfied by the algebra $(A; f)$. Consequently, if two algebras $\mathbb{A}$ and $\mathbb{B}$ generate the same variety, and $\operatorname{Clo} \mathbb{A}$ is minimal, then $\operatorname{Clo} \mathbb{B}$ is minimal, too.

It is convenient to choose a function of the least possible arity as a generator of a minimal clone. These generators are called *minimal functions*: f is a minimal function iff $[f]$ is a minimal clone and there is no nontrivial function in $[f]$ whose arity is less than the arity of f. A minimal function must be of one of five types according to the following theorem of I. G. Rosenberg [Ros] (see also [SzÁ]).

Theorem 1.1. **[Ros]** *Let f be a nontrivial operation of minimum arity in a minimal clone. Then f satisfies one of the following conditions:*

(I) *f is unary, and $f^2(x) = f(x)$ or $f^p(x) = x$ for some prime p;*

(II) *f is a binary idempotent operation, i.e. $f(x, x) = x$;*

(III) *f is a ternary majority operation, i.e. $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$;*

(IV) *$f(x, y, z) = x + y + z$, where $+$ is a Boolean group operation;*

(V) *f is a semiprojection, i.e. there exists an index i $(1 \leq i \leq n)$ such that $f(x_1, \ldots, x_n) = x_i$ whenever the values of $x_1, \ldots, x_n$ are not pairwise distinct.*

A simple induction argument shows that if f is a semiprojection (majority function) and $g \in [f]$ is a nontrivial function of the same arity as f, then g is also a semiprojection (majority function) [Cs2, Qu2, Ros]. Therefore a minimal clone cannot contain minimal functions of two different types, thus we can speak about the five types of minimal clones. We will call a clone generated by an idempotent binary operation a binary clone, and by a majority clone we mean a clone generated by a majority operation.

In cases (I) and (IV) the conditions ensure the minimality of f, while in the other three cases they do not, and a general characterization seems to be far beyond reach. There are numerous partial results that describe minimal clones under certain restrictions; we will discuss some of these in Section 1.4.

Next we state and prove a very special property of clones generated by a majority operation. It is well-known that algebras with a majority operation have many pleasant properties, e.g. they generate congruence distributive varieties, their term functions are determined just by the binary invariant relations (Baker-Pixley theorem), etc. The following theorem shows that majority operations behave very nicely from the viewpoint of minimal clones, too. This fact seems to be folklore; usually it is considered as a consequence of Rosenberg's theorem (or of Świerczkowski's lemma [Sw], which is the starting point in the proof of Rosenberg's theorem). Here we give an almost self-contained proof.

Theorem 1.2. [Cs3] *Let $\mathcal{C}$ be a clone generated by a majority operation f. If every majority operation in $\mathcal{C}$ generates f, then $\mathcal{C}$ is a minimal clone.*

Proof. The key is the following observation, which can be proved by a simple induction argument [Cs3]. If g is a nontrivial operation in a clone generated by a majority function, then g is a so-called *near-unanimity function*, i.e. it satisfies the identities

$$g(y,x,x,\ldots,x,x) = g(x,y,x,\ldots,x,x) = \cdots = g(x,x,x,\ldots,x,y) = x.$$

We show that any near-unanimity function g of arity $n \geq 4$ produces a nontrivial function of arity $n-1$. Let us suppose that $g(x,x,x_3,\ldots,x_n)$ is a projection. Identifying all the x_is except for x_n with x, we get the projection x by the near-unanimity property, therefore $g(x,x,x_3,\ldots,x_n)$ cannot be a projection onto x_n. This can be done for any x_i instead of x_n, thus $g(x,x,x_3,\ldots,x_n)$ has to be a projection onto x. A similar argument shows that if $g(x_1,x_2,y,y,x_5,\ldots,x_n)$ is a projection, then it is a projection onto y. Now we have a contradiction, because $g(x,x,y,y,x_5,\ldots,x_n)$ is a projection to x and y at the same time (this is where we use that $n \geq 4$). Thus we have proved that either $g(x,x,x_3,\ldots,x_n)$ or $g(x_1,x_2,y,y,x_5,\ldots,x_n)$ is nontrivial.
Therefore if g is an at least quaternary near-unanimity function in the clone $\mathcal{C}$, then it produces a nontrivial function of arity one less, which is again a near-unanimity function, since it is still generated by f. Hence if it is still of arity at least 4, then it produces a near-unanimity function of lesser arity, and we can continue this way until we end up with a near-unanimity function of arity 3, i.e. a majority operation. Since it was supposed that every majority operation in $\mathcal{C}$ generates f, we have $f \in [g]$, and this shows that $\mathcal{C}$ is a minimal clone. □

The advantage of this property is that in order to prove the minimality of a clone of type (III) it suffices to prove (1.3) for ternary functions g. On a finite set this means a finite number of functions, while in the binary and semiprojection case one has to consider infinitely many functions.

Another nice property of majority operations is that it is easy to decide whether a composition is trivial or not. If f is a majority function and t is a term of type f, such that for any subterm $f(t_1,t_2,t_3)$ the three arguments t_1, t_2 and t_3 of f

are different, then t^f is not a projection (see [Cs3], where such terms were called regular).

The following lemma is rather obvious, but it may be still worth formulating, as it provides the main tools we will use to prove that a function is not minimal.

Lemma 1.3. *Let f be a function on a set A.*

(i) *If a nontrivial $g \in [f]$ preserves some relation that is not preserved by f, then $[f]$ is not minimal.*

(ii) *If f is a minimal function preserving some $B \subseteq A$, then $f|_B$ must be a minimal or trivial function on B.*

Proof.

(i) Combine (1.3) and (1.1).

(ii) Composing functions and restricting functions commute. □

The second statement of this lemma and the relationship between minimal clones and varieties mentioned in connection with (1.4) are better understood from the viewpoint of abstract clones.

1.2 Abstract clones

An *(abstract) clone* $\mathcal{C}$ is given by a family $\mathcal{C}^{(n)}$ $(n \geq 1)$ of sets with distinguished elements $e_i^{(n)} \in \mathcal{C}^{(n)}$ $(1 \leq i \leq n)$ and mappings

$$F_k^n : \mathcal{C}^{(n)} \times \left(\mathcal{C}^{(k)}\right)^n \to \mathcal{C}^{(k)},\ (f, g_1, \ldots, g_n) \mapsto f\left(g_1, \ldots, g_n\right) \quad (n, k \geq 1),$$

such that the following three axioms are satisfied for all $f \in \mathcal{C}^{(n)}$, $g_1, \ldots, g_n \in \mathcal{C}^{(k)}$, $h_1, \ldots, h_k \in \mathcal{C}^{(l)}$ $(n, k, l \geq 1)$:

$$\begin{aligned} e_i^{(n)}\left(g_1, \ldots, g_n\right) &= g_i \quad (i = 1, \ldots, n); \\ f\left(e_1^{(n)}, \ldots, e_n^{(n)}\right) &= f; \\ f\left(g_1, \ldots, g_n\right)\left(h_1, \ldots, h_k\right) &= f\left(g_1\left(h_1, \ldots, h_k\right), \ldots, g_n\left(h_1, \ldots, h_k\right)\right). \end{aligned}$$

Thus abstract clones are multi-sorted or heterogeneous algebras that capture the compositional structure of concrete clones (considering the elements $e_i^{(n)}$ as nullary operations) [BL, Tay]. The notion of a subclone, clone homomorphism and factor clone can be defined in a natural way, and the isomorphism theorems can be proved for abstract clones.

Every concrete clone can be regarded as an abstract clone if we let $e_i^{(n)}$ be the i-th n-ary projection, and $F_k^n(f, g_1, \ldots, g_n)$ be the composition of f by $g_1, \ldots, g_n$, as we have already indicated it in the notation. We will call the elements $e_i^{(n)}$ projections, the mappings F_k^n composition operations, and $\mathcal{C}^{(n)}$ the n-ary part of $\mathcal{C}$, even if the elements of the abstract clone are not functions. Every abstract clone is isomorphic to a concrete clone, so we can always assume that the elements of the clone are actually functions. This statement is a generalisation of the Cayley representation theorem for monoids, since the unary part of a concrete clone is a transformation monoid, and the defining axioms of abstract clones imply that $\left(\mathcal{C}^{(1)}; F_1^1, e_1^{(1)}\right)$ is a monoid. In the following we will not always make a sharp distinction between concrete and abstract clones.

There is a close relationship between abstract clones and varieties; roughly speaking, abstract clones are the same as varieties up to term equivalence [Kea,LP]. To explain this more explicitly, let us fix an abstract clone $\mathcal{C}$, and a generating set F of $\mathcal{C}$. Then every element $g \in \mathcal{C}$ is obtained from the elements of F and projections by a finite number of compositions. These iterated compositions can be described with the help of terms, just as in the concrete case: $g \in [F]$ iff there is a term of type F such that $g = t^F$. Of course the same element may be obtained by different terms, and the set of pairs $\vartheta = \left\{(t_1, t_2) : t_1^F = t_2^F\right\}$ is an equational theory. The clone is determined up to isomorphism by F and ϑ, so we can say that an abstract clone with a distinguished generating set F carries the same information as an equational theory, i.e. a variety of type F.

In order to describe this variety more explicitly, we need the notion of an F-representation. If $\varphi : \mathcal{C} \to \mathcal{O}_A$, $f \mapsto f^*$ is a clone homomorphism, then we say that the concrete clone $\mathcal{C}^* = \varphi(\mathcal{C})$ is a *representation* of the abstract clone $\mathcal{C}$. Let $[F] = \mathcal{C}$ as before, and let F^* be the image of F under φ. Then $[F^*] = \mathcal{C}^*$, thus we obtain an algebra $\mathbb{A} = (A; F^*)$ of type F with $\mathrm{Clo}(\mathbb{A}) = \mathcal{C}^*$, called an *$F$-representation* of $\mathcal{C}$. Clearly φ is uniquely determined by its restriction to F: if $g = t^F \in \mathcal{C}$, then $g^* = \left(t^F\right)^* = t^{F^*} \in \mathcal{C}^*$. If $(t_1, t_2) \in \vartheta$, then $t_1^F = t_2^F$, therefore $t_1^{F^*} = t_2^{F^*}$, thus $\mathbb{A}$ satisfies the identity $t_1 = t_2$. Conversely, let $\mathbb{A} = (A; F^*)$ be any algebra in the variety defined by ϑ. Then the map $\varphi : \mathcal{C} \to \mathrm{Clo}\,\mathbb{A}$, $t^F \mapsto t^{F^*}$ is a well-defined surjective clone homomorphism: if $t_1^F = t_2^F \in \mathcal{C}$, then $t_1^{F^*} = t_2^{F^*}$, since $\mathbb{A}$ satisfies the identity $t_1 = t_2$, as it belongs to ϑ.

Thus the variety defined by ϑ consists of the F-representations of $\mathcal{C}$, and an algebra $\mathbb{A}$ of type F belongs to this variety if and only if $\mathrm{Clo}\,\mathbb{A}$ is a homomorphic image of $\mathcal{C}$. Since the generating set F is usually clear from the context, we will denote this variety by $\mathcal{V}_\mathcal{C}$. (If we choose another set of generators, then we get another variety which is term-equivalent to the previous one.) Conversely, a clone can be assigned to every variety, namely the clone of the countably generated free algebra of the variety: $\mathrm{Clo}\,\mathcal{V} = \mathrm{Clo}\,\mathbb{F}_{\aleph_0}(\mathcal{V})$, and the maps $\mathcal{C} \mapsto \mathcal{V}_\mathcal{C}$ and $\mathcal{V} \mapsto \mathrm{Clo}\,\mathcal{V}$

are inverses of each other (up to isomorphism of clones, and term-equivalence of varieties).

Let $\mathcal{C}$ and $\mathcal{V}$ correspond to each other at this assignment. Then subvarieties of $\mathcal{V}$ correspond to factor clones of $\mathcal{C}$, and the congruence lattice of $\mathcal{C}$ is dually isomorphic to the subvariety lattice of $\mathcal{V}$. If two algebras $\mathbb{A}, \mathbb{B} \in \mathcal{V}$ generate the same subvariety $\mathcal{W} \leq \mathcal{V}$, then $\operatorname{Clo} \mathbb{A}$ and $\operatorname{Clo} \mathbb{B}$ are isomorphic, since both are isomorphic to $\operatorname{Clo} \mathcal{W}$. Hence if $\operatorname{Clo} \mathbb{A}$ is a minimal clone, then so is $\operatorname{Clo} \mathbb{B}$, as we have already noticed in connection with (1.4). We can also explain the somewhat vague proof of Lemma 1.3 (ii) more precisely now: If $\mathbb{B}$ is a subalgebra of the algebra $\mathbb{A}$, then $\varphi : \operatorname{Clo} \mathbb{A} \to \operatorname{Clo} \mathbb{B}$, $f \mapsto f|_B$ is a surjective clone homomorphism. This is a special case of the fact that $\mathbb{B} \in \mathrm{HSP}(\mathbb{A})$ implies that $\operatorname{Clo} \mathbb{B}$ is a homomorphic image of $\operatorname{Clo} \mathbb{A}$.

The elements of $\mathcal{C}^{(n)}$ may be identified with ϑ-classes of n-ary terms, i.e. with the elements of $\mathbb{F}_n(\mathcal{V})$, the n-generated free algebra of $\mathcal{V}$. Projections correspond to variables under this identification, therefore we will use the notation $x_1, \ldots, x_n$ instead of $e_1^{(n)}, \ldots, e_n^{(n)}$. In the binary case we will also use x and y instead of $e_1^{(2)}$ and $e_2^{(2)}$, and x, y, z will stand for the three ternary projections.

In accordance with the concrete case, an abstract clone is called trivial if it consists of projections only. It is an easy exercise to show that if $e_i^{(m)} = e_j^{(m)}$ holds in a clone $\mathcal{C}$ for some $1 \leq i \neq j \leq m$, then $\left|\mathcal{C}^{(n)}\right| = 1$ for every $n \geq 1$. Therefore there are two trivial clones up to isomorphism: the clone of trivial operations on a set with at least two elements, and the clone of operations on a one-element set. They will be denoted by $\mathcal{I}$ and $\mathcal{I}_1$, respectively; we have $\left|\mathcal{I}^{(n)}\right| = n$ and $\left|\mathcal{I}_1^{(n)}\right| = 1$ for all n. Note that $\mathcal{I}$ plays the role of the smallest clone (up to isomorphism, of course): every clone except $\mathcal{I}_1$ has $\mathcal{I}$ as a subclone (but $\mathcal{I}_1$ cannot be a subclone of any clone other than itself). For quotients the situation is almost the converse: every clone has $\mathcal{I}_1$ as a factor clone (and only some clones have $\mathcal{I}$ as a factor clone). We say that a variety is trivial if its clone is trivial, i.e. isomorphic to $\mathcal{I}$ or $\mathcal{I}_1$ (the usual definition permits only $\mathcal{I}_1$).

An abstract clone is said to be minimal if its only proper subclone is the trivial one (isomorphic to $\mathcal{I}$). Everything we mentioned about minimal clones in the previous section holds almost verbatim in the abstract case (except for those statements that refer explicitly to the underlying set, of course). Subclones of factor clones are always factor clones of subclones, therefore a factor clone of a minimal clone is either minimal or trivial. Consequently, every algebra in a variety with a minimal clone has a minimal or trivial clone. The converse is also true: if a variety consists of algebras with minimal or trivial clones, and there is at least one nontrivial algebra among them, then the variety has a minimal clone.

Absorption identities are very useful in the study of varieties with a minimal clone. These are identities of the form $t = x_i$, i.e. identities with a single variable on one side, and a nontrivial term on the other. The following lemma appears in [LP] and [Kea]; here we present the proof given by P. P. Pálfy and L. Lévai, which uses abstract clones.

Lemma 1.4. [Kea, LP] *Let $\mathcal{V}$ be a variety with a minimal clone, and let $\mathbb{A} \in \mathcal{V}$ be a nontrivial algebra. Then $\mathcal{V}$ satisfies every absorption identity that holds in $\mathbb{A}$.*

Proof. Let $\mathbb{F}_{\aleph_0}(\mathcal{V}) = (T; F)$ be the countably generated free algebra in $\mathcal{V}$, and let $\mathbb{A} = (A; F^*)$. Then $\varphi : \operatorname{Clo} \mathcal{V} \to \operatorname{Clo} \mathbb{A}$, $t^F \mapsto t^{F^*}$ is a surjective clone homomorphism. Since $\mathbb{A}$ is nontrivial, $\mathcal{I}_A$ is a proper subclone of $\operatorname{Clo} \mathbb{A}$, hence $\varphi^{-1}(\mathcal{I}_A)$ is a proper subclone of $\operatorname{Clo} \mathcal{V}$. This latter clone is minimal, therefore the inverse image of $\mathcal{I}_A$ has to be its trivial subclone: $\varphi^{-1}(\mathcal{I}_A) = \mathcal{I}_T$. This equality is exactly what we need; it means that a term interpretes as a projection in $\mathbb{A}$ if and only if it is a projection in $\mathbb{F}_{\aleph_0}(\mathcal{V})$, i.e. if it is a projection in every algebra of $\mathcal{V}$. $\square$

This lemma is particularly useful if the algebra $\mathbb{A}$ is axiomatizable by absorption identities, for in this case we can conclude that $\mathbb{A}$ generates $\mathcal{V}$. We will see some examples of such algebras in the next section.

1.3 Examples

First let us consider the binary case, i.e. clones of idempotent groupoids. (In this dissertation the term *groupoid* refers to an algebra with a single binary operation.) The basic operation of a groupoid will be denoted by $f(x, y) = xy$, and by the *dual* of $\mathbb{A} = (A; f)$ we mean the groupoid $\mathbb{A}^d = (A; f^d)$ with $f^d(x, y) = f(y, x) = yx$. Similarly, $\mathcal{V}^d$ denotes the variety formed by the duals of the elements of $\mathcal{V}$ for a groupoid variety $\mathcal{V}$. Obviously, a groupoid has a minimal clone if and only if its dual does (actually they have the very same clone).

The simplest examples of groupoids (or varieties) with a minimal clone are semilattices and rectangular bands. We list the defining identities of some more groupoid varieties with a minimal clone in Table 1. To save parentheses we write $\overleftarrow{x_1 \cdot \ldots \cdot x_n}$ for the left-associated product $(\cdots((x_1x_2)\,x_3)\cdots)\,x_n$, and similarly $\overrightarrow{x_1 \cdot \ldots \cdot x_n}$ for the right-associated product $x_1(\cdots(x_{n-2}(x_{n-1}x_n))\cdots)$. We abbreviate $\overleftarrow{x \cdot y \cdot \ldots \cdot y}$ to xy^n (where n is certainly the number of y's appearing in the product). Analogously ${}^n xy$ stands for $\overrightarrow{x \cdot \ldots \cdot x \cdot y}$. (We have omitted the identity $xx = x$ everywhere, but of course these are all idempotent varieties.)

The varieties $\mathcal{SL}$ and $\mathcal{RB}$ are selfdual; the duals of right normal bands, right regular bands, right semilattices are left normal bands ($\mathcal{LNB}$), left regular bands ($\mathcal{LRB}$), left semilattices ($\mathcal{LSL}$), respectively. The variety $\mathcal{A}$ is defined by the identity $x(y(zu)) = x((yz)u)$; we will need it later, for the study of almost associative operations. The definition of $\mathcal{D}$ involves infinitely many identities, but

$\mathcal{D} \cap \mathcal{A}$ has the finite basis shown in the table. Indeed, it is quite straightforward to check that any algebra satisfying these identities belongs to $\mathcal{D} \cap \mathcal{A}$. Conversely, if $\mathbb{A} \in \mathcal{D} \cap \mathcal{A}$, then $\mathbb{A} \models x(yz) = x((yy)z) = x(y(yz)) = xy$, and $\mathbb{A}$ also satisfies $xx = x$ and $(xy)y = xy$ as they are among the defining identities of $\mathcal{D}$. This axiomatization of $\mathcal{D} \cap \mathcal{A}$ shows that $\mathcal{D} \cap \mathcal{A}$ contains the variety of right semilattices.

Figure 2 shows the meet-semilattice generated by these varieties and their duals ($\mathcal{LZ}$ and $\mathcal{RZ}$ denote the variety of left and right zero semigroups, and the bottom element is the variety of one-element groupoids). The solid lines indicate covers, while dashed lines connect varieties with some intermediate varieties between them. Note that there is just one $\mathcal{C}_p$ on the picture, but it represents an infinite family of varieties (one for each prime number); we have $\mathcal{C}_{p_1} \cap \mathcal{C}_{p_2} = \mathcal{LZ}$ if $p_1 \neq p_2$.

The minimality of the clone of $\mathcal{B}$ and $\mathcal{D}$ is proved in [LP]; these are the clones in parts (c) and (d) of Theorem 5.2. (Their clone appear in [P^3] as B and $M(2)$.) Both clones contain only two nontrivial binary operations (which are the duals of each other), and every nontrivial operation of higher arity produces these by a suitable identification of variables. J. Płonka introduced p-cyclic groupoids in [Pł2], and he showed that $\mathrm{Clo}\,\mathcal{C}_p$ is minimal iff p is a prime [Pł1]. From now on we will always assume that p denotes a prime number whenever we mention p-cyclic groupoids.

We have not defined the varieties $\mathcal{A}(\mathbb{Z}_p, \lambda)$ appearing in Figure 2 yet. An *affine space* is an algebra whose base set is a vector space over some field, and its clone is the full idempotent reduct of the clone of that vector space. The clone of an affine space is determined by the base field (up to isomorphism), and it is a minimal clone iff this field is isomorphic to $\mathbb{Z}_p$ for some prime number p. If $p = 2$, then this clone is of type (IV): the minority operation $x + y + z$ is a generator of minimum arity. If $p > 2$, then the clone is of type (II): any nontrivial operation of the form $f(x, y) = \lambda x + (1 - \lambda)y$ is a generator. Fixing a $\lambda \in \mathbb{Z}_p \setminus \{0, 1\}$ we get the variety $\mathcal{A}(\mathbb{Z}_p, \lambda)$ of f-representations; it is the variety of groupoids of the form $(V; f)$, where V is a vector space over $\mathbb{Z}_p$, and $f(x, y) = \lambda x + (1 - \lambda)y$. Just as for p-cyclic groupoids, we have indicated these varieties with just two points on Figure 2 (the dual of $\mathcal{A}(\mathbb{Z}_p, \lambda)$ is $\mathcal{A}(\mathbb{Z}_p, 1 - \lambda)$). In this dissertation affine spaces are always meant to be affine spaces over $\mathbb{Z}_p$ (for an arbitrary prime p).

Affine spaces, p-cyclic groupoids and rectangular bands are axiomatizable by absorption identities (cf. [Kea, LP]), therefore we have the following consequence of Lemma 1.4.

Lemma 1.5. [Kea, LP] *Let $\mathcal{V}$ be a variety with a minimal clone, and suppose that $\mathcal{V}$ contains a p-cyclic groupoid (rectangular band, affine space) with a nontrivial clone. Then $\mathcal{V}$ is the variety of p-cyclic groupoids (rectangular bands, affine spaces).*

Proof. Lemma 1.4 shows that $\mathcal{V}$ is *a* variety of p-cyclic groupoids (rectangular bands, affine spaces). However, these varieties have no nontrivial subvarieties, as we can see on Figure 2 (see [LP] or [Kea] for a proof), hence $\mathcal{V}$ is indeed *the* variety of p-cyclic groupoids (rectangular bands, affine spaces). ☐

Let us recall another theorem from [LP] which states that the variety $\mathcal{D}$ is determined by its 2-variable identities and the fact that it has a minimal clone.

Lemma 1.6. [LP] *Let $\mathcal{V}$ be a variety with a minimal clone satisfying the identities $x\,(yx) = (xy)\,x = (xy)\,y = (xy)\,(yx) = xy, x\,(xy) = x$. Then $\mathcal{V}$ is a subvariety of $\mathcal{D}$.*

Proof. This is part (d) of Theorem 5.2 in [LP]. The identities listed here are sufficient to determine the two-generated free algebra of $\mathcal{V}$. Its multiplication table is the following (the four elements have to be distinct, since otherwise $\mathrm{Clo}\,(\mathcal{V})$ would be trivial).

$\cdot$	x	y	xy	yx
x	x	xy	x	xy
y	yx	y	yx	y
xy	xy	xy	xy	xy
yx	yx	yx	yx	yx

It is not hard to check that this groupoid satisfies every identity of the form $x \cdot \overleftarrow{x \cdot y_1 \cdot \ldots \cdot y_n} = x$ (this is a special case of Lemma 4.2 in [LP]). These are absorption identities, therefore we can apply Lemma 1.4 with $\mathbb{A} = \mathbb{F}_2\,(\mathcal{V})$ to show that $\mathcal{V}$ satisfies these identities, too. The remaining identities in the definition of $\mathcal{D}$ are the same as the ones that were assumed. ☐

There are much less examples of minimal clones of type (III). The simplest ones are those containing just one nontrivial ternary operation (these are all minimal by Theorem 1.2). An example of such a clone is the clone generated by the median function $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ on an arbitrary lattice [PK].

There is no minimal clone with exactly two majority functions (see Theorem 3.6), so the next simplest examples are those that contain three majority functions. The dual discriminator function [FP] on any set A defined by

$$d\,(a,b,c) = \begin{cases} a & \text{if } a = b \\ c & \text{if } a \neq b \end{cases}$$

generates only three majority functions: itself, $d\,(y,z,x)$ and $d\,(z,x,y)$; any of these clearly generates d, therefore $[d]$ is a minimal clone by Theorem 1.2 (cf. [CsG]).

We will see some more examples of minimal majority functions in the next section.

1.4 Characterizations

It seems to be a very hard problem to characterize minimal clones in full generality, but there are some results that describe minimal clones or minimal functions under certain assumptions. In this section we mention some of these results; we formulate precisely only the theorems that we will need in the sequel.

One of the first, and most natural approaches is to restrict the size of the underlying set of a concrete clone. E. Post determined all clones on the two-element set [Po]; seven of them are minimal. Minimal clones on the three-element set were described by B. Csákány [Cs2]. For the four-element set minimal clones of type (II) were described by B. Szczepara [Szcz]. A nontrivial semiprojection on a four-element set has to be of arity 3 or 4, and the latter case was settled in [JQ]. We are going to describe minimal majority functions on the four-element set in Chapter 2; the case of ternary semiprojections remains open. We will need the list of the minimal majority functions on the three-element set, so let us state this theorem.

Theorem 1.7. **[Cs2]** *If f is a minimal majority function on a three-element set, then f is isomorphic to one of the the twelve majority functions shown in Table* 3. *These functions belong to three minimal clones containing* $1, 3$ *and* 8 *majority operations respectively, as shown in the table.*

Note that we have omitted those triples in the table where the majority rule determines the value of the functions. The nicest generators of the three clones are m_1, m_2 and m_3. We see that m_1 is a very simple function; it is as constant as a majority function can be. It can be defined also as the median function of the three element chain (with the unusual order $2 < 1 < 3$ or $3 < 1 < 2$). The next function looks like the first projection, and it is nothing else but the dual discriminator, up to a permutation of variables (the third function in $[m_2]$ is actually the dual discriminator). The function m_3 follows a nice pattern as well, but it can be described by formulas better than words (for completeness we define m_1 and m_2 formally, too). For $\{a_1, a_2, a_3\} = \{1, 2, 3\}$ we have

$$
\begin{aligned}
&m_1(a_1, a_2, a_3) = 1;\\
&m_2(a_1, a_2, a_3) = a_1;\\
&m_3(a_1, a_2, a_3) = a_{i+1} \quad \text{if } a_i = 2 \text{ (subscripts taken modulo 3).}
\end{aligned}
$$

One may restrict the size of the clone instead of the underlying set as well. There is a result in this direction by L. Lévai and P. P. Pálfy; they described binary minimal clones with at most seven binary operations [LP]. (The cases 5 and 7 are actually due to J. Dudek and J. Gałuszka, cf. [Du, DG].) In Chapter 3 we are going to characterize minimal majority clones with at most seven ternary operations. Here we quote only the list of binary clones with at most four binary operations (i.e. at most two nontrivial binary operations).

Theorem 1.8. [LP] *Let $\mathcal{V}$ be a groupoid variety with a minimal clone such that* $\operatorname{Clo}\mathcal{V}$ *contains at most four binary operations. Then $\mathcal{V}$ is a subvariety of one of the varieties* $\mathcal{A}(\mathbb{Z}_3, 2), \mathcal{B}, \mathcal{C}_2, \mathcal{D}, \mathcal{RB}$ *or the variety defined by* $(xy)\,x = (xy)\,y = (xy)\,(yx) = xy$ *and* $x \cdot \overleftarrow{y \cdot x \cdot z_1 \cdot \ldots \cdot z_n} = x \ \ (n = 0, 1, 2, \ldots)$.

It is a possibility to make some assumptions on the relations preserved by a function. Considering unary relations, an extremal situation is the case of algebras with no nontrivial subalgebras; such algebras having a minimal clone were described by B. Csákány and K. Kearnes [CsK]. Conservative operations are on the other extreme: a function is *conservative* if it preserves every subset of the underlying set (cf. [Qu1]). Conservative minimal binary and majority operations were described by B. Csákány [Cs3]. J. Ježek and R. W. Quackenbush obtained results for conservative semiprojections, for example minimal n-ary semiprojections on the n-element set (they are necessarily conservative) are known [JQ].

Before we state the theorem about conservative minimal majority operations, let us make an observation and introduce some notation. For any set A let $\binom{A}{3}$ denote the set of three-element subsets of A. If f is a conservative minimal majority function on A, and $B \in \binom{A}{3}$, then $f|_B$ is a minimal majority function on B by Lemma 1.3(ii). These restrictions determine f, so we can say that f is somehow glued together from copies of the functions listed in Table 3; let $f\|_B$ be the one of these 12 functions for which $(B; f|_B) \cong (\{1, 2, 3\}\,; f\|_B)$ holds. There are many ways to do this gluing, and only a few of them yield minimal functions; the next theorem describes exactly which ones.

Theorem 1.9. [Cs3] *A conservative majority function f on a set A is minimal iff its restriction to any three-element subset is minimal, and $[f\|_{B_1}] = [f\|_{B_2}]$ implies $f\|_{B_1} = f\|_{B_2}$ for all three-element subsets B_1, B_2, i.e. if at most one function appears from each of the clones $[m_2]$ and $[m_3]$ as a restriction of f. If f is such a function and $g \in [f]$ is a majority function, then we have*

$$\forall B \in \binom{A}{3} : [g\|_B] = [f\|_B]\,;$$
$$\forall B_1, B_2 \in \binom{A}{3} : \Big([g\|_{B_1}] = [g\|_{B_2}] \implies g\|_{B_1} = g\|_{B_2}\Big).$$

Every map $B \mapsto g\|_B$ satisfying the above two properties appears for exactly one function $g \in [f]$. Therefore the number of majority functions in $[f]$ is either $1, 3, 8$ *or* 24.

Remark 1.10. It is useful to consider restrictions to three-element subsets even for majority functions that are not conservative. The proof of the previous theorem gives the following necessary condition for the minimality of a majority function f on A preserving $B_1, B_2 \in \binom{A}{3}$:

$$[f\|_{B_1}] = [f\|_{B_2}] \implies f\|_{B_1} = f\|_{B_2}.$$

Another possibility is to look for minimal functions satisfying certain identities. Probably the most natural problem of this kind is to characterize semigroups with a minimal clone. This problem was solved by M. B. Szendrei; she determined all bands whose subclone lattice is a chain [SzM]. Here we reproduce the proof given by P. P. Pálfy in [P^3].

Theorem 1.11. [P^3, SzM] *A semigroup with a minimal clone is either a left regular band, a right regular band or a rectangular band.*

Proof. Let f be an idempotent associative binary operation generating a minimal clone. It is a straightforward calculation to check that the operation $g(x,y) = xyx$ does not generate any other binary operation but itself and its dual. If g is nontrivial, then $f \in [g]$ by (1.3), therefore $xyx = xy$ or $xyx = yx$ holds, thus we have a left or right regular band. If g is the second projection, then $xyx = y$, and this implies $xy = (xy)(xy) = (xyx)y = yy = y$, a contradiction. Finally, if g is the first projection, then we have $xyx = x$, consequently $xyz = (xzx)y(zxz) = (xz)(xyz)(xz) = xz$, so our semigroup is a rectangular band. $\square$

In Chapter 5 we will generalize this theorem by characterizing minimal clones generated by almost associative binary operations for two different interpretations of the term 'almost associative'.

Á. Szendrei and K. Kearnes investigated minimal clones generated by an operation that commutes with itself [KSz]. In the binary case this commutativity property is equivalent to the so-called entropic or medial law $(xy)(zu) = (xz)(yu)$, and the result is the following.

Theorem 1.12. [KSz] *Let $\mathbb{A}$ be an entropic groupoid with a minimal clone. Then $\mathbb{A}$ or its dual is an affine space, a rectangular band, a left normal band, a right semilattice or a p-cyclic groupoid for some prime p.*

We will show in Theorem 4.4 that we get the same list of minimal clones if we assume only distributivity (which is weaker than entropicity for idempotent groupoids). We will also characterize groupoids satisfying the identity $x(yz) = xy$ and having a minimal clone (cf. Lemma 4.8).

Finally, let us quote a result of K. Kearnes describing abelian algebras with a minimal clone [Kea]. It turns out that such clones are either of type (I), (II) or (IV), and in the binary case the entropic law holds. Therefore the following theorem could be deduced from the previous one, but actually Theorem 1.12 was proved with the help of Theorem 1.13. In Chapter 4 we generalize this theorem to weakly abelian representations.

Theorem 1.13. [**Kea**] *If a minimal clone has a nontrivial abelian representation, then it is either unary, or the clone of an affine space, a rectangular band or a p-cyclic groupoid for some prime p.*

Chapter 2

Minimal majority clones on the four-element set

In this chapter we determine the minimal majority functions on the four-element set. The main result is the following theorem which characterizes nonconservative minimal majority operations on the set $\{1, 2, 3, 4\}$. (The conservative ones are already described in Theorem 1.9.)

Theorem 2.20 **[Wa1]** *If f is a minimal majority function on the set $\{1, 2, 3, 4\}$, then f is either conservative, or isomorphic to one of the twelve majority functions shown in Table 4. These functions belong to three minimal clones containing $1, 3$ and 8 majority operations respectively, as shown in the table. Moreover, the clone generated by M_i is isomorphic to $[m_i]$ (see Table 3) for $i = 1, 2, 3$.*

In Section 2.1 we make some observations that will show that we do not have to consider all the 4^{24} majority operations on $\{1, 2, 3, 4\}$, only those that satisfy a certain identity. The next three sections contain the proof of the above theorem. The hard part of the proof is to show that the above twelve functions are the only minimal majority operations on the four-element set. We divide the set of majority functions under consideration into two classes: 'orderly' and 'disorderly' functions. In Section 2.2 we prove that every minimal disorderly function is isomorphic to M_2, and in Section 2.3 we show that up to isomorphism and permutation of variables M_1 and M_3 are the only orderly minimal functions. In Section 2.4 we prove that the clones generated by m_i and M_i are isomorphic, hence the latter are minimal functions.

2.1 Majority functions on finite sets

To find all minimal majority operations on a given finite set is a finite task according to Theorem 1.2. However, on a four-element set there are 4^{24} majority functions, so it seems hopeless to test them one by one, even using a computer. The next theorem reduces this number by showing that it suffices to consider functions satisfying a certain identity.

Theorem 2.1. **[Wa1]** *Let f be a majority function on a finite set A. Then there exists a majority function $g \in [f]$ which satisfies the following identity.*

$$g\big(\, g(x,y,z)\,,\, g(y,z,x)\,,\, g(z,x,y)\,\big) = g(x,y,z) \tag{2.1}$$

Proof. Let us define a binary operation on $[f]^{(3)}$ by the following formula.

$$(g * h)\,(x,y,z) = g\big(\, h(x,y,z)\,,\, h(y,z,x)\,,\, h(z,x,y)\,\big)$$

It follows from the associativity of composition (see the third axiom in the definition of abstract clones) that this operation is associative. The set of majority functions is closed under this operation, so $\Big([f]^{(3)} \setminus \mathcal{I}_A; *\Big)$ is a finite semigroup. Every finite semigroup has an idempotent element (moreover, every element has an idempotent power), and if g is an idempotent in our semigroup, then it satisfies (2.1). □

Now we introduce some notation. The k-th power $f * \cdots * f$ of f will be denoted by $f^{(k)}$, and $\widehat{f}$ denotes an idempotent power of f (whose existence is guaranteed by the above theorem). We put $\langle abc \rangle = \{(a,b,c),(b,c,a),(c,a,b)\}$, and we will use the symbol $f|_{\langle abc \rangle} \equiv u$ to mean that $f(a,b,c) = f(b,c,a) = f(c,a,b) = u$, and $f|_{\langle abc \rangle} = p$ to mean that $f(a,b,c) = a, f(b,c,a) = b, f(c,a,b) = c$. (Here '$p$' stands for 'projection': $f|_{\langle abc \rangle} = p$ means that f agrees with the first projection on the set $\langle abc \rangle$. If both $f|_{\langle abc \rangle} = p$ and $f|_{\langle bac \rangle} = p$ hold, then $f|_{\{a,b,c\}}$ looks like a first projection – except that it is a majority function. Similarly, $f|_{\langle abc \rangle} \equiv u \equiv f|_{\langle bac \rangle}$ means that f is as constant on $\{a,b,c\}$ as a majority function can be.) The following lemma shows an easy way to test if a majority function satisfies identity (2.1) or not.

Lemma 2.2. **[Wa1]** *Let f be a majority function on a set A satisfying* (2.1)*, and let a,b,c be pairwise distinct elements of A. Let $u = f(a,b,c)$, $v = f(b,c,a)$, $w = f(c,a,b)$. Then $|\{u,v,w\}| \neq 2$, and if u,v,w are pairwise different, then $f|_{\langle uvw \rangle} = p$.*

Proof. To prove the first statement, let us suppose (without loss of generality) that $u \neq v = w$. Then (2.1) for $x = a,\ y = b,\ z = c$ yields that $f(u,v,w) = u$, contradicting the majority property of f. The second statement of the lemma is an obvious consequence of identity (2.1). □

We can say a bit more than Lemma 2.2 when f is a minimal function.

Theorem 2.3. [Wa1] *Let f be a minimal majority function on a set A satisfying* (2.1)*, and let a, b, c be pairwise distinct elements of A. If $u = f(a, b, c)$, $v = f(b, c, a)$, $w = f(c, a, b)$ are pairwise different, then $f|_{\langle uvw\rangle} = p$ and also $f|_{\langle vuw\rangle} = p$.*

Proof. By the previous lemma we have $f|_{\langle uvw\rangle} = p$. Now the nontrivial superposition $g(x, y, z) = f(f(x, y, z), f(x, z, y), x)$ preserves $\{u, v, w\}$ hence f does too, and then from the description of the minimal majority functions on the three-element set (Theorem 1.7) we get the conclusion of the theorem. □

In the next lemma we consider the four-element case. Let us recall that $\langle abc\rangle$ is just the set $\{(a, b, c),\ (b, c, a),\ (c, a, b)\}$, hence $f(\langle abc\rangle)$ denotes $\{f(a, b, c),\ f(b, c, a),\ f(c, a, b)\}$.

Lemma 2.4. [Wa1] *Let f be a minimal majority function on the four-element set $A = \{a, b, c, d\}$ satisfying* (2.1)*. If $f(\langle abc\rangle) \subseteq \{a, b, c\}$ then either $f|_{\langle abc\rangle} = p$ and $f|_{\langle bac\rangle} = p$ or $f|_{\langle abc\rangle} \equiv u$ and $f|_{\langle bac\rangle} \equiv v$ for some $u, v \in A$.*

Proof. The set $f(\langle abc\rangle)$ has either three or one elements by Lemma 2.2. If it has three elements then it is $\{a, b, c\}$, and then by Theorem 2.3 we have $f|_{\langle abc\rangle} = p$ and $f|_{\langle bac\rangle} = p$. If $f(\langle abc\rangle)$ is a one-element set, then we may assume $f|_{\langle abc\rangle} \equiv a$. If $d \notin f(\langle bac\rangle)$, then f preserves $\{a, b, c\}$ and Theorem 1.7 shows that $f|_{\langle bac\rangle} \equiv v$ for suitable $v \in \{a, b, c\}$. If $a, d \in f(\langle bac\rangle)$ then we permute cyclically the variables to have $f(b, a, c) = a$, and then $g^{(2)}$ preserves $\{a, b, c\}$ for the superposition g in the proof of Theorem 2.3, contradicting the minimality of f. Finally, if $a \notin f(\langle bac\rangle)$ but $d \in f(\langle bac\rangle)$ then $f(\langle bac\rangle) = \{b, c, d\}$. Now we may suppose $f(b, a, c) = c$, $f(a, c, b) = d$, $f(c, b, a) = b$ or $f(b, a, c) = b$, $f(a, c, b) = d$, $f(c, b, a) = c$ after a cyclic permutation of variables. In both cases $g^{(2)}$ shows that f is not minimal, since it preserves $\{a, b, c\}$. □

If f is a conservative minimal majority function on a set A satisfying (2.1), then for all $a, b, c \in A$ we have

$$f|_{\langle abc\rangle} = p \text{ or } (\exists u \in A) : f|_{\langle abc\rangle} \equiv u. \tag{2.2}$$

This follows from Theorem 1.9, but it can be deduced from Lemma 2.2 and Theorem 2.3 as well. Lemma 2.4 and Theorem 2.3 leave such an impression that (2.2) holds for many $a, b, c \in A$ even in the nonconservative case, if A has just four elements. Let us call a majority function f on $\{1, 2, 3, 4\}$ satisfying (2.1) 'orderly' if (2.2) is valid for all a, b, c and 'disorderly' otherwise. In the next section we will prove that up to isomorphism there is only one minimal disorderly function on a four-element set, namely M_2. In Section 2.3 we investigate orderly functions, and we will find that the only minimal ones are M_1 and M_3 up to isomorphism and permutation of variables.

Now we define and examine a superposition which we will use frequently later on. For a ternary function f let f_x, f_y, f_z stand for the composite functions where the first, second resp. third variable of f is replaced by f itself:

$$\begin{aligned} f_x(x,y,z) &= f(f(x,y,z),y,z);\\ f_y(x,y,z) &= f(x,f(x,y,z),z);\\ f_z(x,y,z) &= f(x,y,f(x,y,z)). \end{aligned}$$

We will briefly write f_{zy} instead of $(f_z)_y$. We will also use the convention that lower indices have priority to upper ones. So $f_{zy}^{(k)}$ means $(f_{zy})^{(k)}$, and $\widehat{f}_{zy}$ stands for $\widehat{(f_{zy})}$.

The proof of the following claim is just a straightforward calculation, so we omit it.

Claim 2.5. *Let f be a majority function on $\{a,b,c,d\}$. If $f(a,b,c) \neq d$, then $f_{zy}(a,b,c) = f(a,b,c)$. If $f(a,b,c) = d$, then $f_{zy}(a,b,c) = f(a,b,d)$ if the latter does not equal d. If $f(a,b,c) = f(a,b,d) = d$, then $f_{zy}(a,b,c) = f(a,d,c)$ if this value is not b. Finally, if $f(a,b,c) = f(a,b,d) = d$ and $f(a,d,c) = b$, then $f_{zy}(a,b,c) = f(a,d,b)$.*

The following six lines summarize the statement of this claim in the case $\{a,b,c,d\} = \{1,2,3,4\}$ and $d = 4$, which we will consider most of the time.

$$\begin{aligned}
f_{zy}(1,2,3) &= f(1,2,3) \xrightarrow{4} f(1,2,4) \xrightarrow{4} f(1,4,3) \xrightarrow{2} f(1,4,2)\\
f_{zy}(2,3,1) &= f(2,3,1) \xrightarrow{4} f(2,3,4) \xrightarrow{4} f(2,4,1) \xrightarrow{3} f(2,4,3)\\
f_{zy}(3,1,2) &= f(3,1,2) \xrightarrow{4} f(3,1,4) \xrightarrow{4} f(3,4,2) \xrightarrow{1} f(3,4,1)\\
f_{zy}(2,1,3) &= f(2,1,3) \xrightarrow{4} f(2,1,4) \xrightarrow{4} f(2,4,3) \xrightarrow{1} f(2,4,1)\\
f_{zy}(1,3,2) &= f(1,3,2) \xrightarrow{4} f(1,3,4) \xrightarrow{4} f(1,4,2) \xrightarrow{3} f(1,4,3)\\
f_{zy}(3,2,1) &= f(3,2,1) \xrightarrow{4} f(3,2,4) \xrightarrow{4} f(3,4,1) \xrightarrow{2} f(3,4,2)
\end{aligned}$$

An arrow of the form $u \xrightarrow{w} v$ indicates that we need to compute u, and we can stop here, if $u \neq w$; while if we find that $u = w$, then we have to compute v (and follow the next arrow similarly, if there is one). For example, in order to find $f_{zy}(1,2,3)$ we compute first $f(1,2,3)$. If this value is not 4, then we are done: $f_{zy}(1,2,3) = f(1,2,3)$. If $f(1,2,3) = 4$, then we have to proceed to $f(1,2,4)$. If it is not 4, then we can stop: $f_{zy}(1,2,3) = f(1,2,4)$; otherwise we need to go on to $f(1,4,3)$, and we are done if it equals 2. If $f(1,4,3) = 2$, then $f_{zy}(1,2,3) = f(1,4,2)$. We will consider f_{zy} very often, and we will not refer to Claim 2.5 all the time; the reader should always look at the above table (or make a similar one) when we claim anything about the values of f_{zy}.

In the following two sections we will scan through the disorderly and orderly functions on $\{1,2,3,4\}$, and check that almost all of them are not minimal. Most often this will be done with the help of Lemma 1.3 (i) by finding a nontrivial superposition which preserves some subset that is not preserved by the original function. The following lemma presents another tool for proving the nonminimality of a function. This lemma was proved in [Cs2] by term induction; here we give a proof using invariant relations, but first we need a definition. Majority functions are obviously surjective, therefore it is more meaningful to define the *range* of a majority operation as follows. If f is a majority function on a set A, then let

$$\operatorname{range}(f) = \{f(a,b,c) : a,b,c \in A \text{ are pairwise distinct}\}.$$

Lemma 2.6. **[Cs2]** *If f is a majority function on A and $g \in [f]^{(3)} \setminus \mathcal{I}_A$, then* $\operatorname{range}(g) \subseteq \operatorname{range}(f)$. *Moreover, if f is a minimal majority function, then we have* $\operatorname{range}(g) = \operatorname{range}(f)$.

Proof. It is easy to check that a majority operation preserves the equivalence relation whose blocks are $\{a\}$ and $A \setminus \{a\}$ if and only if a does not belong to its range. Now the first statement follows from (1.1), the second one from (1.3). □

2.2 Disorderly functions

In this section we will show that every disorderly minimal function is isomorphic to M_2. Since we will consider the values of the functions on the set $\{1,2,3\}$ very often, it will be useful to introduce the following notation. Let $[p,q,r;s,t,u]$ denote the set of majority functions f on $A = \{1,2,3,4\}$ for which $f(1,2,3) = p$, $f(2,3,1) = q$, $f(3,1,2) = r$, $f(2,1,3) = s$, $f(1,3,2) = t$, $f(3,2,1) = u$ holds. If we do not want to specify all these six values of f, than we will use $*$ to indicate an arbitrary element of A. For example $f \in [4,*,*;*,*,*]$ means just that $f(1,2,3) = 4$. The letters a,b,c,d will always denote arbitrary distinct elements of A, i.e. $\{1,2,3,4\} = A = \{a,b,c,d\}$.

Claim 2.7. *In either of the following four cases f is not minimal.*

(1) $f \in [4,2,1;*,*,*]$

(2) $f \in [4,1,2;*,*,*]$

(3) $f \in [4,1,3;*,*,*]$

(4) $f \in [4,3,1;*,*,*]$

Proof.

(1) Suppose for contradiction that $f \in [4,2,1;*,*,*]$ is a minimal function. Then we have $f|_{\langle 214\rangle} = p = f|_{\langle 124\rangle}$ by Theorem 2.3, and this implies $f_{zy}|_{\langle 214\rangle} = p = f_{zy}|_{\langle 124\rangle}$. Using Claim 2.5 we can check that $f_{zy} \in [1,2,1;u,v,w]$ with $u,v \neq 4$. Since f does not preserve $\{1,2,3\}$, we must have $w = 4$. A more careful analysis shows that this happens only if $f(3,2,1) = f(3,2,4) = f(3,4,1) = 4$, or $f(3,2,1) = f(3,2,4) = f(3,4,2) = 4$ and $(3,4,1) = 2$. Let us examine the set $f(\langle 213\rangle)$ now. It has either one or three elements by Lemma 2.2, and it is not a subset of $\{1,2,3\}$ according to Lemma 2.4. Therefore we have $f(\langle 213\rangle) = \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$ or $\{4\}$. We treat these four cases separately.

Case 1. If $f(\langle 213\rangle) = \{1,2,4\}$, then $f \in [4,2,1;1,2,4] \cup [4,2,1;2,1,4]$ since $f(3,2,1) = 4$. Using the fact that $f_{zy}|_{\langle 214\rangle} = p = f_{zy}|_{\langle 124\rangle}$ we conclude that $f_{zy} \in [1,2,1;1,2,4] \cup [1,2,1;2,1,4]$ and $\widehat{f_{zy}} \in [1,1,1;1,2,4] \cup [1,1,1;2,1,4]$. However, in this case Lemma 2.4 shows that $\widehat{f_{zy}}$ is not minimal, hence neither is f.

Case 2. If $f(\langle 213\rangle) = \{1,3,4\}$, then $f|_{\langle 134\rangle} = p = f|_{\langle 314\rangle}$ by Theorem 2.3 contradicting that $f(3,4,1)$ is either 4 or 2.

Case 3. If $f(\langle 213\rangle) = \{2,3,4\}$, then similarly to the previous case we have $f|_{\langle 234\rangle} = p = f|_{\langle 324\rangle}$ contradicting that $f(3,2,4) = 4$.

Case 4. If $f(\langle 213\rangle) = \{4\}$, then $f_{zy} \in [1,2,1;2,v,4]$ with $v \neq 4$. As we have already seen in the first case, $v = 1$ is not possible. We cannot have $v = 2$ either, because this would imply $f_{zy}^{(2)} \in [1,1,1;2,2,2]$, hence $f_{zy}^{(2)}$ would preserve $\{1,2,3\}$ contradicting the minimality of f. Finally, if $v = 3$ then $f_{zy}(y,z,f_{zy}(x,y,z)) \in [2,1,1;3,3,2]$, which is a contradiction again.

(2) Here we can use the same argument; the only difference is that in this case $f_{zy} \in [1,1,2;u,v,w]$ with $u,v \neq 4$.

(3) The function $f(x,z,y)$ is isomorphic to a function which is not minimal by (1). (We shall note here that interchanging the second and third variable does not affect the identity (2.1).)

(4) Now $f(x,z,y)$ falls under (2) after renaming the elements of the base set. $\square$

Claim 2.8. *If $f \in [4,3,2;*,*,*]$ then f is not minimal.*

Proof. Suppose for contradiction that $f \in [4,3,2;*,*,*]$ is a minimal function. Similarly to the previous claim we have $f|_{\langle 234\rangle} = p = f|_{\langle 324\rangle}$, and the four possibilities for $f(\langle 213\rangle)$ are $\{1,2,4\}, \{1,3,4\}, \{2,3,4\}$ and $\{4\}$.

Case 1. If $f(\langle 213\rangle) = \{1,2,4\}$, then $f|_{\langle 214\rangle} = p = f|_{\langle 124\rangle}$ by Theorem 2.3. Now Claim 2.5 shows that $f_{zy} \in [1,3,2;u,v,w]$ with $u,v,w \neq 4$, hence f_{zy} preserves $\{1,2,3\}$, which is a contradiction.

Case 2. If $f(\langle 213\rangle) = \{1,3,4\}$, then $f|_{\langle 134\rangle} = p = f|_{\langle 314\rangle}$, and we get the same contradiciton as in the previous case.

Case 3. If $f(\langle 213\rangle) = \{2,3,4\}$, then $f \in [4,3,2;3,4,2]$ or $f \in [4,3,2;2,4,3]$, as otherwise f would be isomorphic to a function which is not minimal by Claim 2.7. Now we have $g \in [3,3,2;3,2,2]$ or $g \in [3,3,2;2,2,4]$ for the function $g(x,y,z) = f(z,y,f(x,y,z))$, and we get a contradiction, because in both cases $g^{(2)} \in [3,3,3;2,2,2]$, and thus $g^{(2)}$ preserves $\{1,2,3\}$.

Case 4. If $f(\langle 213\rangle) = \{4\}$, then $f \in [4,3,2;4,4,4]$ and $g \in [3,3,2;u,2,v]$, where g is the same function as above. If $u \neq 3$, then for $h(x,y,z) = g(g(x,y,z),z,x)$ we have $h \in [3,2,2;2,2,*]$. (In order to verify this for $u = 4$ one needs to observe that $f|_{\langle 234\rangle} = p = f|_{\langle 324\rangle}$ implies $g(4,3,2) = 2$.) Now we have a contradiction, as $h^{(2)} \in [2,2,2;2,2,2]$. So let us suppose that $u = 3$ and $v = 4$ (otherwise g preserves $\{1,2,3\}$). This means that $g \in [3,3,2;3,2,4]$, and one can check that $\widehat{g} \in [3,3,3;3,2,4]$ or $\widehat{g} \in [3,3,3;2,4,3]$ or $\widehat{g} \in [3,3,3;4,3,2]$. (Again, we need the fact that $g(4,3,2) = 2$, $g(3,2,4) = 4$ and $g(2,4,3) = 3$ follows from $f|_{\langle 234\rangle} = p = f|_{\langle 324\rangle}$.) Lemma 2.4 shows that $\widehat{g}$ is not a minimal function, hence neither is f. $\square$

Claim 2.9. *If $f \in [4,2,3;2,1,4]$ or $f \in [4,2,3;4,1,3]$ then f is not minimal.*

Proof. If $f \in [4,2,3;2,1,4]$ is a minimal function, then $f|_{\langle 234\rangle} = p = f|_{\langle 324\rangle}$ and $f|_{\langle 214\rangle} = p = f|_{\langle 124\rangle}$ by Theorem 2.3. This implies that $f_z \in [1,2,3;2,1,3]$, thus f_z preserves $\{1,2,3\}$, and we have a contradiction. The second case is similar; here we have $f_y \in [1,2,3;2,1,3]$. $\square$

Claim 2.10. *If $f \in [4,2,3;2,4,3]$ is a minimal function, then $f = M_2$.*

Proof. Let us consider the function $g(x,y,z) = f(f(x,y,z),x,y)$. Then we have $g \in [u,2,3;2,v,3]$, where $u = f(4,1,2)$ and $v = f(4,1,3)$. We also have $g|_{\langle 234\rangle} = p = g|_{\langle 324\rangle}$ since $f|_{\langle 234\rangle} = p = f|_{\langle 324\rangle}$ follows from Theorem 2.3. If none of u and v equals 4, then g preserves $\{1,2,3\}$, which is impossible. If $u = 4$ and $v = 1$, $v = 2$ or $v = 3$, then $\widehat{g} \in [4,2,3;2,1,3]$, $\widehat{g} \in [4,2,3;2,2,2]$ or $\widehat{g} \in [4,2,3;3,3,3]$ respectively. In either case $\widehat{g}$ is not minimal by Lemma 2.4. If $u \neq 4 = v$, then we get a contradiction in a similar way. Therefore we must have $f(4,1,2) = f(4,1,3) = 4$. Using $g(x,y,z) = f(f(x,y,z),y,x)$, $g(x,y,z) = f(x,f(x,y,z),y)$, etc. we get $f(4,2,1) = f(4,3,1) = 4$, $f(1,4,2) = f(1,4,3) = 4$, etc. Thus we obtained $f|_{\langle 124\rangle} \equiv 4 \equiv f|_{\langle 214\rangle}$ and $f|_{\langle 134\rangle} \equiv 4 \equiv f|_{\langle 314\rangle}$, and taking into account that $f|_{\langle 234\rangle} = p = f|_{\langle 324\rangle}$ we conclude $f = M_2$. $\square$

Claim 2.11. *If $f \in [4,2,3;4,4,4]$ then f is not minimal.*

Proof. Let $f \in [4,2,3;4,4,4]$ be a minimal function. Just like in the previous claim, we have $f|_{\langle 234\rangle} = p = f|_{\langle 324\rangle}$, and this implies $f_{zy}|_{\langle 234\rangle} = p = f_{zy}|_{\langle 324\rangle}$. Therefore $f_{zy} \in [u,2,3;v,w,3]$ with $v \neq 4$.

Case 1. If $v = 3$, then $f_{zy} \in [u,2,3;3,w,3]$, therefore $\widehat{f_{zy}} \in [1,2,3;3,3,3]$, $\widehat{f_{zy}} \in [2,2,2;3,3,3]$, $\widehat{f_{zy}} \in [3,3,3;3,3,3]$ or $\widehat{f_{zy}} \in [4,2,3;3,3,3]$ depending on whether $u = 1,2,3$ or 4. We have a contradiction, because in the first three cases $\widehat{f_{zy}}$ preserves $\{1,2,3\}$, while in the last case $\widehat{f_{zy}}$ is not minimal by Lemma 2.4.

Case 2. If $v = 1$, then for the function $h(x,y,z) = f_{zy}(z,x,f_{zy}(x,y,z))$ we have $h \in [*,2,3;3,*,3]$. The same argument as above leads to a contradiction, since $\widehat{h}$ either preserves $\{1,2,3\}$ or is not minimal by Lemma 2.4.

Case 3. If $v = 2$, then we consider f_{zy} again. At least one of u and w must equal 4, as otherwise f_{zy} preserves $\{1,2,3\}$. If $u = 4 \neq w$, then $\widehat{f_{zy}} \in [4,2,3;2,1,3]$, $\widehat{f_{zy}} \in [4,2,3;2,2,2]$, $\widehat{f_{zy}} \in [4,2,3;3,3,3]$ depending on whether $u = 1,2$ or 3. Thus we have a contradiction, because $\widehat{f_{zy}}$ is not minimal by Lemma 2.4. The case $u \neq 4 = w$ is similar, so let us suppose $u = v = 4$. Then $f_{zy} \in [4,2,3;2,4,3]$, hence $f_{zy} = M_2$ according to Claim 2.10. We will see later that the clone generated by M_2 does not contain any function belonging to $[4,2,3;4,4,4]$ (cf. Theorem 2.20), therefore $f \notin [f_{zy}]$ contradicting the minimality of f. □

Theorem 2.12. [**Wa1**] *Every disorderly minimal majority function on the set $A = \{1,2,3,4\}$ is isomorphic to M_2.*

Proof. Claim 2.7 and Claim 2.8 together with Lemma 2.4 show that if f is a minimal majority function on A satisfying (2.1) and $f(\langle abc\rangle)$ is a three-element set but $f|_{\langle abc\rangle} \neq p$, then on two of the triplets $(a,b,c), (b,c,a), (c,a,b)$ the value of f equals the first variable, while on the third one f equals d.
If f is disorderly, then this happens for some $a,b,c \in A$, and we can suppose without loss of generality that $\langle abc\rangle = \langle 123\rangle$, and $f(1,2,3) = 4$. Therefore $f \in [4,2,3;u,v,w]$ for some $u,v,w \in A$, and we cannot have $u = v = w = 4$ by Claim 2.11. Now Lemma 2.4 yields that $f(\langle 213\rangle)$ has three elements and $f|_{\langle 213\rangle} \neq p$. Thus we can apply the argument of the previous paragraph with $\langle abc\rangle = \langle 213\rangle$ and we conclude that $f \in [4,2,3;2,1,4]$ or $f \in [4,2,3;4,1,3]$ or $f \in [4,2,3;2,4,3]$. (Note that after fixing $f(1,2,3) = 4$ it would restrict the generality if we assumed, say, that $u = 4$.) The first two cases are not possible by Claim 2.9, while in the third case Claim 2.10 shows that f equals M_2. □

2.3 Orderly functions

In this section we are going to search for the orderly minimal functions. The conservative ones are already described, so we deal only with nonconservative functions. We assume f to be such a function and we will prove several properties of f, until we find that only a few functions possess these properties, namely M_1, M_3 and $M_3(y,x,z)$ (up to isomorphism).

So let f be an arbitrary nonconservative orderly minimal majority function on $A = \{1,2,3,4\}$. It follows from Lemma 2.4 that a stronger form of (2.2) is valid: for any three-element subset $\{a,b,c\}$ of A either $f|_{\langle abc\rangle} = p$ and $f|_{\langle bac\rangle} = p$ or $f|_{\langle abc\rangle} \equiv u$ and $f|_{\langle bac\rangle} \equiv v$ holds for some $u, v \in A$. If the latter happens for all four three-element subsets, then f is invariant under cyclic permutations of its variables, i.e. it is *cyclically symmetric*. In the first claim we show that our function f has to be cyclically symmetric.

Claim 2.13. *The function f is cyclically symmetric.*

Proof. Suppose that f is a orderly nonconservative minimal function that is not cyclically symmetric. Then there are $a,b,c \in A$ such that $f|_{\langle abc\rangle} = p = f|_{\langle bac\rangle}$, say $f|_{\langle 124\rangle} = p = f|_{\langle 214\rangle}$. Since f is not conservative we may suppose that $f|_{\langle 123\rangle} \equiv u$ and $f|_{\langle 213\rangle} \equiv v$ where at least one of u and v equals 4. If $u \neq 4$ then $f_{zy}|_{\langle 123\rangle} \equiv u$, while if $u = 4$ then $f_{zy}(1,2,3) = 1$ and $f_{zy}(2,3,1) \neq 4$.
Thus we have $f_{zy}(\langle 123\rangle) \subseteq \{1,2,3\}$ except when $u = 4$ and $f_{zy}(3,1,2) = 4$. Claim 2.5 shows that the latter holds only if $f(3,1,2) = f(3,1,4) = f(3,4,1) = 4$ and $f(3,4,2) = 1$, or $f(3,1,2) = f(3,1,4) = f(3,4,2) = 4$. In the first case $f(3,4,2) = 1$ implies $f|_{\langle 234\rangle} \equiv 1$ since f is orderly, and therefore we have $f_{zy} \in [1,1,4;*,*,*]$.
Similarly, in the second case we have $f|_{\langle 234\rangle} \equiv 4$ and this implies $f_{zy} \in [1,2,4;*,*,*]$. However, $f_{zy} \in [1,2,4;*,*,*]$ leads to a contradiction as follows. Since $f|_{\langle 124\rangle} = p = f|_{\langle 214\rangle}$ we have $f_{zy}|_{\langle 124\rangle} = p = f_{zy}|_{\langle 214\rangle}$ as well, and therefore $\widehat{f_{zy}} \in [1,2,4;*,*,*]$. We see that $\widehat{f_{zy}}$ is disorderly, thus Theorem 2.12 implies that $\widehat{f_{zy}}$ is isomorphic to M_2. We will see in the proof of Theorem 2.20 that the clone generated by M_2 contains no orderly functions, hence $f \notin [\widehat{f_{zy}}]$ contradicting the minimality of f.
We have proved that either $f_{zy}(\langle 123\rangle) \subseteq \{1,2,3\}$ or $f_{zy} \in [1,1,4;*,*,*]$, and similarly one can verify that $f_{zy}(\langle 213\rangle) \subseteq \{1,2,3\}$ or $f_{zy} \in [*,*,*;2,2,4]$. Combining these possibilities we get the following four cases.

Case 1. If $f_{zy}(\langle 123\rangle) \subseteq \{1,2,3\}$ and $f_{zy}(\langle 213\rangle) \subseteq \{1,2,3\}$, then f_{zy} preserves $\{1,2,3\}$, which is a contradiction.

Case 2. If $f_{zy}(\langle 123\rangle) \subseteq \{1,2,3\}$ and $f_{zy} \in [*,*,*;2,2,4]$, then we have a contradiction again, because $\widehat{f_{zy}}$ preserves $\{1,2,3\}$. To verify this let us suppose that

$f_{zy} \in [r,s,t;2,2,4]$ where $r,s,t \neq 4$. If r,s,t are not pairwise distinct, say, $r=s$, then $\widehat{f_{zy}} \in [r,r,r;2,2,2]$, hence $\widehat{f_{zy}}$ preserves $\{1,2,3\}$. If $\{r,s,t\} = \{1,2,3\}$, then we have two possibilities: either $\langle rst\rangle = \langle 123\rangle$ or $\langle rst\rangle = \langle 213\rangle$. In the first case $\widehat{f_{zy}} \in [1,2,3;2,2,2]$, while in the second case $\widehat{f_{zy}} \in [2,2,2;2,2,2]$, therefore in both cases $\widehat{f_{zy}}$ preserves $\{1,2,3\}$.

Case 3. If $f_{zy} \in [1,1,4;*,*,*]$ and $f_{zy}(\langle 213\rangle) \subseteq \{1,2,3\}$, then a similar argument leads to a contradiction.

Case 4. If $f_{zy} \in [1,1,4;*,*,*]$ and $f_{zy} \in [*,*,*;2,2,4]$, then clearly we have $\widehat{f_{zy}} \in [1,1,1;2,2,2]$, a contradiction again. □

From now on we suppose f to be a nonconservative cyclically symmetric minimal majority function on A. In [Csi] these are determined by computer, here we give a straightforward description. Since f is not conservative, we can suppose without loss of generality that $f|_{\langle 123\rangle} \equiv 4$. In the following two claims we prove that f preserves all three-element subsets of A except for $\{1,2,3\}$.

Claim 2.14. *If $f|_{\langle 123\rangle} \equiv 4$ and $f|_{\langle 213\rangle} \equiv u \neq 4$, then the only subset of A not preserved by f is $\{1,2,3\}$.*

Proof. Suppose for contradiction that f does not preserve, say, $\{1,2,4\}$. Then we have $f|_{\langle 124\rangle} \equiv 3$ or $f|_{\langle 214\rangle} \equiv 3$ or both. First let us assume that $f|_{\langle 124\rangle} \equiv 3$, and let us consider the function $g(x,y,z) = f(x, f_z(x,y,z), z)$. If $f(2,3,4) = 4$, then $g \in [3,3,*;u,u,u]$, thus $g^{(2)}$ preserves $\{1,2,3\}$ contradicting the minimality of f. So we have $f(2,3,4) \neq 4$, and this implies $f_{zy} \in [3,v,w;u,u,u]$ with $v \neq 4$. Since f does not preserve $\{1,2,3\}$, we must have $w = 4$. Claim 2.5 shows that this holds only if $f(3,1,4) = f(3,4,1) = 4$ and $f(2,3,4) = 1$. However, this implies that $g \in [3,1,1;u,u,u]$, which is a contradiction again.
Now let us suppose that $f|_{\langle 214\rangle} \equiv 3$ and $f|_{\langle 124\rangle} \equiv v \neq 3$, and let $h(x,y,z) = f(y,x,f(x,y,z))$. Then $h \in [3, f(3,2,4), f(1,3,4); u_1,u_2,u_3]$, where at least two of u_1,u_2,u_3 equals u. Now we separate six cases upon the value of $f(3,2,4)$ and $f(1,3,4)$.

Case 1. If $f(3,2,4) = 3$ or $f(1,3,4) = 3$, then $h^{(2)}$ preserves $\{1,2,3\}$ since $h^{(2)} \in [3,3,3;u,u,u]$.

Case 2. If $f(3,2,4) = f(1,3,4) = 1$, then $h^{(2)}$ preserves $\{1,2,3\}$ again, as $h^{(2)} \in [1,1,1;u,u,u]$.

Case 3. If $f(3,2,4) = f(1,3,4) = 2$, then similarly to the previous case we have $h^{(2)} \in [2,2,2;u,u,u]$.

Case 4. If $f(3,2,4) = 1$ and $f(1,3,4) = 2$, then one can check that $h^{(2)} \in [2,3,1;u,u,u]$, which is a contradiction again.

Case 5. If $f(3,2,4) = 2$ and $f(1,3,4) = 1$, then $h^{(3)}$ preserves $\{1,2,3\}$, because $h^{(2)} \in [u_3,u_1,u_2;u,u,u]$ and thus $h^{(3)} \in [u,u,u;u,u,u]$.

Case 6. If $f(3,2,4) = 4$ or $f(1,3,4) = 4$, then let us consider the values of h on $\{1,2,4\}$. We have $h(2,1,4) = 4,\ h(1,4,2) = f(1,3,4),\ h(4,2,1) = f(3,2,4)$ and $h(1,2,4) = v_1,\ h(2,4,1) = v_2,\ h(4,1,2) = v_3$, where at least two of v_1,v_2,v_3 equals v. Therefore $h^{(2)}|_{\langle 214\rangle} \equiv 4$ and $h^{(2)}|_{\langle 124\rangle} \equiv v \neq 3$, hence $h^{(2)}$ preserves $\{1,2,4\}$. This is a contradiction, as f does not preserve $\{1,2,4\}$. $\square$

Claim 2.15. *If $f|_{\langle 123\rangle} \equiv 4$ and $f|_{\langle 213\rangle} \equiv 4$, then the only subset of A not preserved by f is $\{1,2,3\}$.*

Proof. Let us suppose again that f does not preserve $\{1,2,4\}$. By the previous claim we must have $f|_{\langle 124\rangle} \equiv 3 \equiv f|_{\langle 214\rangle}$. Since f is cyclically symmetric $f|_{\langle 234\rangle} \equiv u_1, f|_{\langle 324\rangle} \equiv v_1, f|_{\langle 314\rangle} \equiv u_2, f|_{\langle 134\rangle} \equiv v_2$ with suitable $u_1,v_1,u_2,v_2 \in A$. Let us now examine the values of f_{zy} on $\{1,2,3\}$. Taking into account that $f|_{\langle 124\rangle} \equiv 3 \equiv f|_{\langle 214\rangle}$ we can simplify the table following Claim 2.5 in the following way (see the left column).

$$\begin{array}{ll}
f_{zy}(1,2,3) = 3 & f_{zy}(1,2,4) = 4 \\
f_{zy}(2,3,1) = u_1 \xrightarrow{4} 3 & f_{zy}(2,4,1) = v_1 \xrightarrow{3} 4 \\
f_{zy}(3,1,2) = u_2 \xrightarrow{4} u_1 \xrightarrow{1} v_2 & f_{zy}(4,1,2) = v_2 \xrightarrow{3} v_1 \xrightarrow{1} u_2 \\
f_{zy}(2,1,3) = 3 & f_{zy}(2,1,4) = 4 \\
f_{zy}(1,3,2) = v_2 \xrightarrow{4} 3 & f_{zy}(1,4,2) = u_2 \xrightarrow{3} 4 \\
f_{zy}(3,2,1) = v_1 \xrightarrow{4} v_2 \xrightarrow{2} u_1 & f_{zy}(4,2,1) = u_1 \xrightarrow{3} u_2 \xrightarrow{2} v_1
\end{array}$$

We see that $f_{zy}(2,3,1) \neq 4$, and $f_{zy}(3,1,2) = 4$ iff

(1) $u_2 = u_1 = 4$ or

(2) $u_2 = v_2 = 4$ and $u_1 = 1$.

Similarly $f_{zy}(1,3,2) \neq 4$, and $f_{zy}(3,2,1) = 4$ iff

(3) $v_1 = v_2 = 4$ or

(4) $v_1 = u_1 = 4$ and $v_2 = 2$.

Since f does not preserve $\{1,2,3\}$, at least one of (1)–(4) must hold. The right column of the table shows the values of f_{zy} on $\{1,2,4\}$, and as f does not preserve this set either, $f_{zy}(4,1,2) = 3$ or $f_{zy}(4,2,1) = 3$ holds. Therefore at least one of the following four statements is true:

(5) $v_2 = v_1 = 3$;

(6) $v_2 = u_2 = 3$ and $v_1 = 1$;

(7) $u_1 = u_2 = 3$;

(8) $u_1 = v_1 = 3$ and $u_2 = 2$.

Now we need to consider all the 16 combinations of (1)–(4) and (5)–(8). Fortunately, most of these pairs are not consistent; only (1) and (5), or (3) and (7) can hold simultaneously. In the first case $f_{zy} \in [3,3,4;3,3,3]$; in the second case $f_{zy} \in [3,3,3;3,3,4]$, hence in both cases $f_{zy}^{(2)}$ preserves $\{1,2,3\}$ contradicting the minimality of f. $\square$

We have proved that if f is an orderly nonconservative minimal function, then f is cyclically symmetric and preserves all but one three-element subsets of A. In the following three claims – as usually – we suppose that $f|_{\langle 123\rangle} \equiv 4$, $f|_{\langle 213\rangle} \equiv u$ and f preserves $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$. Depending on whether $u = 4$ or not, we will finally reach M_1 or M_3.

Claim 2.16. *If $f|_{\langle 123\rangle} \equiv 4$, then $\widehat{f_{zy}}(\langle 123\rangle) \subseteq \{1,2,3\}$, unless $f(1,2,4) = f(2,3,4) = f(3,1,4) = 4$.*

Proof. Assume that $f|_{\langle 123\rangle} \equiv 4$, and let $u = f(1,2,4), v = f(2,3,4), w = f(3,1,4)$. First suppose that none of u, v, w equals 4. Then $f_{zy} \in [u,v,w;*,*,*]$, and if u, v, w are not pairwise distinct, say $u = v$, then $\widehat{f_{zy}} \in [u,u,u;*,*,*]$, i.e. $\widehat{f_{zy}}(\langle 123\rangle) = \{u\}$. If $\{u,v,w\} = \{1,2,3\}$, then we have $\langle uvw\rangle = \langle 123\rangle$. Indeed, $(u,v,w) = (2,1,3)$ is impossible, because $v \in \{2,3,4\}$, and (u,v,w) cannot be $(1,3,2)$ or $(3,2,1)$ either, since $w \in \{1,3,4\}$ and $u \in \{1,2,4\}$. Now it is easy to check that $\widehat{f_{zy}}(\langle 123\rangle) = \{1,2,3\}$.
Next suppose that exactly one of u, v, w equals 4, say $u = 4 \neq v, w$. Then Claim 2.5 shows that $f_{zy} \in [w,v,w;*,*,*]$, hence $\widehat{f_{zy}} \in [w,w,w;*,*,*]$, i.e. $\widehat{f_{zy}}(\langle 123\rangle) = \{w\}$. Finally, if two of u, v, w equals 4, say $u, v = 4 \neq w$, then we have $f_{zy} \in [w,4,w;*,*,*]$, therefore $\widehat{f_{zy}} \in [w,w,w;*,*,*]$ holds again. $\square$

Claim 2.17. *If $f|_{\langle 213\rangle} \equiv u \neq 4$ then f is isomorphic to M_3 or $M_3(y,x,z)$.*

Proof. We can assume without loss of generality that $u = 3$. Then $f_{zy}|_{\langle 213\rangle} \equiv 3$, therefore $\widehat{f_{zy}}$ preserves $\{1,2,3\}$ iff $\widehat{f_{zy}}(\langle 123\rangle) \subseteq \{1,2,3\}$. Thus we must have $f(1,2,4) = f(2,3,4) = f(3,1,4) = 4$ by the previous claim. Now f is determined by its values on $\langle 214\rangle, \langle 324\rangle$ and $\langle 134\rangle$. Since f preserves $\{1,2,4\}, \{2,3,4\}$ and

$\{1,3,4\}$ we have three choices for each of these three values. The following table lists the 27 possibilities.

$\langle 214\rangle$	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	4	4	4	4	4	4	4	4	4
$\langle 324\rangle$	3	3	3	2	2	2	4	4	4	3	3	3	2	2	2	4	4	4	3	3	3	2	2	2	4	4	4
$\langle 134\rangle$	1	3	4	1	3	4	1	3	4	1	3	4	1	3	4	1	3	4	1	3	4	1	3	4	1	3	4
	a	a	a	a	b	b	a	b	c	b	a	b	a	a	a	b	a	c	b	a	e	a	b	c	c	d	c

If f corresponds to a column marked with the letter 'a', then f is not minimal, because it does not satisfy the condition in Remark 1.10. For example, let us consider the first column. Here $\left[f\|_{\{1,2,4\}}\right] = \left[f\|_{\{2,3,4\}}\right] = [m_3]$, but $f\|_{\{1,2,4\}} \neq f\|_{\{2,3,4\}}$. (Actually f restricted to $\{1,2,4\}$ is isomorphic to m_3, while the restriction to $\{2,3,4\}$ is isomorphic to $m_3\,(y,x,z)$.)
For functions marked with 'b' let us consider the composition $g(x,y,z) = f(y,x,f(x,y,z))$. We have $g \in [f(2,1,4), f(3,2,4), f(1,3,4); 4,3,3]$, therefore $f(g(x,y,z), g(z,x,y), g(y,z,x)) \in [v,v,v;3,3,3]$ where v stands for $f\,(f(2,1,4), f(1,3,4), f(3,2,4))$. It turns out that $v \neq 4$ in all of the 8 cases, hence $f(g(x,y,z), g(z,x,y), g(y,z,x))$ preserves $\{1,2,3\}$, which is a contradiction. For example, if f corresponds to the last column that is marked with 'b', then $g \in [4,2,3;4,3,3]$ and $v = f\,(4,3,2) = 2$.
For columns marked with 'c' we claim that $3 \notin \operatorname{range}\left(h^{(2)}\right)$, where $h(x,y,z) = f(x, f(x,y,z), f\,(y,x,z))$. Indeed, the range of $f|_{\{1,2,4\}}$, $f|_{\{2,3,4\}}$ and $f|_{\{1,3,4\}}$ does not contain 3, hence the same is true for $h^{(2)}$. Thus it suffices to verify that $3 \notin \operatorname{range}\left(h^{(2)}|_{\{1,2,3\}}\right)$. For the function corresponding to the third column from the right we have $h^{(2)} \in [4,4,4;1,1,1]$; for the other four functions $h^{(2)} \in [4,4,4;4,4,4]$ holds. The range of f contains 3 since we assumed $f|_{\langle 213\rangle} \equiv 3$, and therefore we have a contradiction by Lemma 2.6.
Finally, the function marked with 'd' is isomorphic to M_3, and the one marked with 'e' is $M_3\,(y,x,z)$. $\square$

Claim 2.18. *If $f|_{\langle 213\rangle} \equiv 4$ then $f = M_1$.*

Proof. Let $U = \{f(1,2,4),\ f(3,1,4),\ f(2,3,4)\}$ and $V = \{f(2,1,4),\ f(1,3,4),\ f(3,2,4)\}$. If $U \neq \{4\}$ then $\widehat{f_{zy}}\,(\langle 123\rangle) \subseteq \{1,2,3\}$ by Claim 2.16, and similarly one can verify that $V \neq \{4\}$ implies $\widehat{f_{zy}}\,(\langle 213\rangle) \subseteq \{1,2,3\}$. Since f does not preserve $\{1,2,3\}$ we must have $U = \{4\}$ or $V = \{4\}$. Let us suppose first that $U = \{4\} \neq V$. Then $\widehat{f_{zy}}|_{\langle 123\rangle} \equiv 4$ and $\widehat{f_{zy}}\,(\langle 213\rangle) \subseteq \{1,2,3\}$. Now Lemma 2.4 shows that $\widehat{f_{zy}}|_{\langle 213\rangle} \equiv u \neq 4$, thus $\widehat{f_{zy}}$ satisfies the conditions of the previous claim. Therefore f is isomorphic to a function belonging to the clone generated by M_3. However, there is no function in $[M_3]$ with $f|_{\langle abc\rangle} \equiv d \equiv f|_{\langle bac\rangle}$. Similarly $U = \{4\} \neq V$ is not possible either. Hence we must have $U = \{4\} = V$, and then $f = M_1$. $\square$

Let us summarize the results of this section.

Theorem 2.19. [**Wa1**] *Every nonconservative orderly minimal majority function on* $A = \{1,2,3,4\}$ *is isomorphic to* M_1, M_3 *or* $M_3(y,x,z)$.

2.4 The minimal clones

Theorem 2.20. [**Wa1**] *If* f *is a minimal majority function on the set* $\{1,2,3,4\}$, *then* f *is either conservative, or isomorphic to one of the twelve majority functions shown in Table* 4. *These functions belong to three minimal clones containing* 1, 3 *and* 8 *majority operations respectively, as shown in the table. Moreover, the clone generated by* M_i *is isomorphic to* $[m_i]$ (*see Table* 3) *for* $i = 1,2,3$.

Proof. Theorems 2.12 and 2.19 show that every nonconservative minimal majority operation on $\{1,2,3,4\}$ is isomorphic to a function generated by M_1, M_2 or M_3. It remains to prove that the clones $[m_i]$ and $[M_i]$ are isomorphic for $i = 1,2,3$. This implies that the M_i are indeed minimal functions, and we will also see that they generate exactly the twelve majority operations shown in Table 4. We present two proofs for this isomorphism: an 'abstract' and a 'concrete' one.
The abstract approach is quite easy: it suffices to prove that the algebras $(\{1,2,3\};m_i)$ and $(\{1,2,3,4\};M_i)$ generate the same variety. Clearly the first algebra embeds into the second one (the embedding is $1 \mapsto 4, 2 \mapsto 2, 3 \mapsto 3$). On the other hand, $(\{1,2,3,4\};M_i)$ is isomorphic to a subalgebra of the direct square of $(\{1,2,3\};m_i)$; for example $1 \mapsto (1,2), 2 \mapsto (2,1), 3 \mapsto (3,1), 4 \mapsto (1,1)$ is an embedding.
The concrete proof is more elaborate, but it is constructive: we prove that $f \mapsto f|_{\{2,3,4\}}$ is an isomorphism between $[M_i]$ and $\left[M_i|_{\{2,3,4\}}\right]$, which is isomorphic to $[m_i]$, since the algebras $\left(\{2,3,4\};M_i|_{\{2,3,4\}}\right)$ and $(\{1,2,3\};m_i)$ are isomorphic. It is obvious that this restriction is a surjective clone homomorphism, so it suffices to show that every $f \in [M_i]$ is uniquely determined by its restriction to $\{2,3,4\}$.
Let σ and ϱ be the equivalence relations corresponding to the partitions $\{\{1,4\},\{2\},\{3\}\}$ and $\{\{1\},\{2,3,4\}\}$ respectively. Since M_i preserves these equivalence relations and all unary relations except for $\{1,2,3\}$, any $f \in [M_i]$ also preserves them. There is only one majority operation on a two-element set, therefore the restrictions of M_i to two-element subsets are all isomorphic. Moreover, *any* bijection between two-element subsets is an isomorphism between the corresponding restrictions of M_i, consequently the same is true for every $f \in [M_i]$.
Now let $f_1, f_2 \in [M_i]$ be n-ary operations such that $f_1|_{\{2,3,4\}} = f_2|_{\{2,3,4\}}$, and let $a_1,\ldots,a_n$ be arbitrary elements of $\{1,2,3,4\}$. Our goal is to prove that $f_1(a_1,\ldots,a_n) = f_2(a_1,\ldots,a_n)$.

We define the elements a'_j and a''_j as follows.

$$a'_j = \begin{cases} a_j & \text{if } a_j \neq 1 \\ 4 & \text{if } a_j = 1 \end{cases} \quad a''_j = \begin{cases} 3 & \text{if } a_j \neq 1 \\ 1 & \text{if } a_j = 1 \end{cases} \quad (j = 1, \ldots, n)$$

We have $a_j \sigma a'_j$ and $a'_j \in \{2, 3, 4\}$, therefore

$$f_1 (a_1, \ldots, a_n) \, \sigma f_1 (a'_1, \ldots, a'_n) = f_2 (a'_1, \ldots, a'_n) \, \sigma f_2 (a_1, \ldots, a_n) ,$$

hence $f_1 (a_1, \ldots, a_n) \, \sigma f_2 (a_1, \ldots, a_n)$. If $f_1 (a'_1, \ldots, a'_n) \in \{2, 3\}$, then we are done, because 2 and 3 are singleton blocks of σ. If $f_1 (a'_1, \ldots, a'_n) = 4$, then we can conclude only that $f_1 (a_1, \ldots, a_n), f_2 (a_1, \ldots, a_n) \in \{1, 4\}$. Since 1 and 4 are not related in ϱ, it suffices to show that $f_1 (a_1, \ldots, a_n) \, \varrho f_2 (a_1, \ldots, a_n)$. We have $f_1|_{\{2,3\}} = f_2|_{\{2,3\}}$, and for $k = 1, 2$ the algebras $\left(\{2, 3\} ; f_k|_{\{2,3\}}\right)$ and $\left(\{1, 3\} ; f_k|_{\{1,3\}}\right)$ are isomorphic under the *same* isomorphism (say, $2 \mapsto 1, 3 \mapsto 3$), therefore $f_1|_{\{1,3\}} = f_2|_{\{1,3\}}$. Thus $f_1 (a''_1, \ldots, a''_n) = f_2 (a''_1, \ldots, a''_n)$, and then

$$f_1 (a_1, \ldots, a_n) \, \varrho f_1 (a''_1, \ldots, a''_n) = f_2 (a''_1, \ldots, a''_n) \, \varrho f_2 (a_1, \ldots, a_n) .$$

follows, since $a_j \varrho a''_j$. By transitivity we have $f_1 (a_1, \ldots, a_n) \, \varrho f_2 (a_1, \ldots, a_n)$, and this completes the second proof. In order to find the ternary operations in the clones $[M_i]$ we can use Theorem 1.7 to determine their restrictions to $\{2, 3, 4\}$, and then apply the above argument to extend these restrictions to $\{1, 2, 3, 4\}$. The resulting functions are shown in Table 4. □

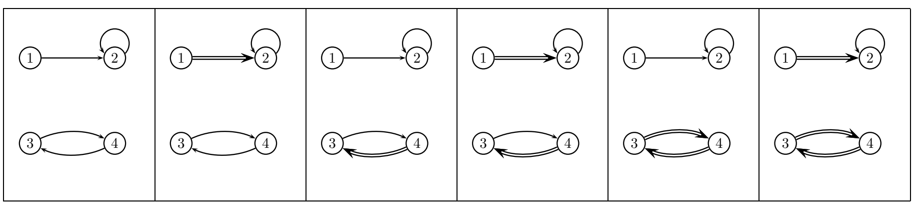
There are $4,\ 12,\ 24$ majority operations on $A = \{1, 2, 3, 4\}$ that are isomorphic to $M_1,\ M_2,\ M_3$ respectively, so there are $4+12+24 = 40$ nonconservative majority minimal clones on a four-element set. These clones contain $4 \cdot 1 + 12 \cdot 3 + 24 \cdot 8 = 232$ majority operations, hence there are 232 nonconservative minimal majority operations on A, and they fall into $1 + 3 + 8 = 12$ isomorphism classes.

For completeness, let us count the conservative clones and operations, too. We know from Theorem 1.9 that every conservative minimal majority clone can be generated by a unique operation whose restrictions to three-element subsets are isomorphic to $m_1,\ m_2$ or m_3. Conversely, every such operation generates a minimal clone, and they generate different clones, therefore it suffices to determine the number of these functions. Let us say that these are our "favourite" generators. On a given three-element set there are $3,\ 1,\ 3$ operations isomorphic to $m_1,\ m_2,\ m_3$ respectively, hence we have $3 + 1 + 3 = 7$ choices on each of the four three-element subsets. Consequently, the number of conservative minimal majority clones on the four-element set is $7^4 = 2401$. (This is easy to generalize: on the n-element set there are $7^{\binom{n}{3}}$ conservative minimal majority clones.)

To count the minimal functions, let us note that Theorem 1.9 shows that if both m_2 and m_3 appear (in an isomorphic copy) among the restrictions of our generator to three-element sets, then the clone contains 24 majority operations. If m_3 appears, but m_2 does not, then we have 8 majority functions in the clone; if m_2 appears, but m_3 does not, then we get 3 majority functions. Finally, if all the restriction to three-element sets are isomorphic to m_1, then the clone contains just one majority operation. Therefore, the number of nonconservative minimal majority operations on A is $3^4 \cdot 1 + (4^4 - 3^4) \cdot 3 + (6^4 - 3^4) \cdot 8 + (7^4 - 6^4 - 4^4 + 3^4) \cdot 24 = 32\,646$. (This also generalizes to arbitrary finite base sets; we leave it to the reader to write up the formula.)

To see how many functions we get if we count only up to isomorphism, we return to our 2401 favourite generators, and assign a directed graph to each of them. The vertex set is $A = \{1, 2, 3, 4\}$, and there will be exactly four edges: one edge leaving from every vertex. In the graph corresponding to the majority operation f, the edge leaving from $a \in A$ is determined by the restriction of f to $\{b, c, d\} = A \setminus \{a\}$ as follows. If this restriction is isomorphic to m_2, then we draw an arrow from a to a (a loop). If $f|_{\{b,c,d\}}$ is isomorphic to m_1, then we draw an arrow from a to b if $f|_{\langle bcd \rangle} \equiv b \equiv f|_{\langle cbd \rangle}$. If $f|_{\{b,c,d\}}$ is isomorphic to m_3, say $f|_{\langle bcd \rangle} \equiv b$ and $f|_{\langle cbd \rangle} \equiv c$, then we draw a double arrow from a to d. Clearly two functions are isomorphic iff the corresponding graphs are isomorphic (regarding single and double edges as different).

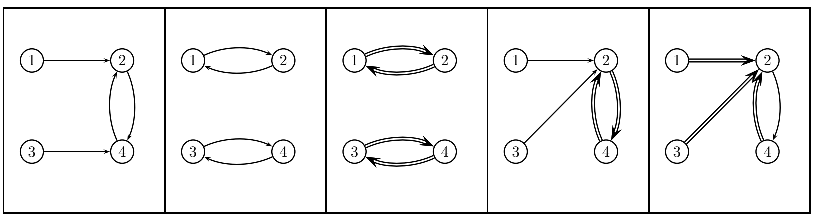
If we do not distinguish between single and double arrows, then we get the graph of a map $A \to A$, and conversely, the graph of any transformation of A is the graph of one of our favourite generators. So we only have to count the number of unary operations on A up to isomorphism, and then consider the possible ways to double some of the arrows in their graphs. The results are summarized in Figure 6. There are 19 graphs with only single arrows, and for each of them we gave the number of ways to double some of the arrows (the first number below each graph). Note that loops are always single arrows, and we have to take into account the symmetries (i.e. automorphisms) of the graph at the counting. For example, consider the graph in the second column of the second row. Here we have three edges that we can double, but the two edges at the bottom play symmetric roles, therefore only 6 of the total number of 2^3 possibilities yield nonisomorphic graphs:



We end up with 126 favourite generators up to isomorphism, i.e. there are 126 conservative minimal majority clones on the four-element set up to *algebra isomorphism*. We can apply Theorem 1.9 again to count the number of majority operations in these clones. If there is a loop and a double arrow, then we get 24 majority functions; if there is a double arrow but no loop, then we get 8; if there is a loop but no double arrow, then we get 3, and if there are only single arrows (none of whom is a loop), then we get only one function. In the example above, we obtain $1 \cdot 3 + 5 \cdot 24 = 123$ functions. Performing this calculation for all of the 19 cases (see the second number below each graph in Figure 6) we find that there are 1653 conservative minimal majority operations on the four element set up to isomorphism.

The hardest task is to count the clones up to *clone isomorphism*. Let f_1 and f_2 be conservative minimal majority operations on A. If the algebras $\mathbb{A}_1 = (A; f_1)$ and $\mathbb{A}_2 = (A; f_2)$ generate the same variety, then the clones $[f_1]$ and $[f_2]$ are isomorphic. Unfortunately, the converse is not true in general: the isomorphism of the clones ensures only that the two varieties are term equivalent. However, if f_1 and f_2 are the favourite generators of the corresponding clones, then the converse holds as well. To prove this, we observe that the favourite generator is canonical in the sense that if f is one of our favourite generators, then it is the only majority function in $[f]$ that satisfies the identities $f^{(2)} = f$ and $g^{(2)} = f$, where g stands for the operation $f(f(x,y,z), f(y,x,z), z)$. This implies that if φ is an isomorphism from $[f_1]$ to $[f_2]$, then $\varphi(f_1) = f_2$, hence $\mathrm{HSP}\,\mathbb{A}_1 = \mathrm{HSP}\,\mathbb{A}_2$.

Therefore we only need to find the varieties generated by algebras of the form $\mathbb{A} = (A; f)$, where f is one of the 126 favourite generators. Any variety is determined by its subdirectly irreducible members, and according to Jónsson's lemma, these are in $\mathrm{HS}\mathbb{A}$ in our case, since $\mathbb{A}$ generates a congruence-distributive variety. As A has only four elements, there is no difficulty in listing all the algebras in $\mathrm{HS}\mathbb{A}$. We omit the details, and present only the final results: 121 of the 126 algebras are subdirectly irreducible (hence they generate pairwise different varieties); only the algebras represented by the following five graphs are not subdirectly irreducible:



For the first two algebras $\mathrm{HS}\mathbb{A}$ contains only one- and two-element subdirectly irreducible algebras; for the other three cases $\mathrm{HS}\mathbb{A}$ contains $(\{1,2,3\}; m_3)$ as well.

Thus there are 123 conservative minimal majority clones on the 4-element set up to clone isomorphism.

We have seen in Theorem 2.20 that there are three nonconservative minimal majority clones on $\{1, 2, 3, 4\}$ up to clone isomorphism. The variety generated by $(\{1, 2, 3, 4\}\,; M_i)$ contains only one- and two-element subdirectly irreducible algebras for $i = 1$, and it contains also $(\{1, 2, 3\}\,; m_i)$ for $i = 2, 3$. Thus the clones $[M_1]$ and $[M_3]$ are isomorphic to some of the 123 conservative ones, but $[M_2]$ is not: altogether there are 124 minimal majority clones on the four-element set up to isomorphism. The numerical outcomes of the above discussion are summarized in Table 5.

Chapter 3

Minimal clones with few majority functions

We will study minimal majority clones as abstract clones in this chapter. Theorem 1.2 shows that the minimality of a majority clone can be read off from its ternary operations, thus it suffices to consider the algebra $\left(\mathcal{C}^{(3)}; F_3^3, e_1^{(3)}, e_2^{(3)}, e_3^{(3)}\right)$ only. We will refer to this algebra as the ternary part of $\mathcal{C}$, and denote it by $\mathcal{C}^{(3)}$. This is an algebra with one quaternary and three nullary operations satisfying the following identities.

$$\begin{aligned} F_3^3\left(e_i^{(3)}, f_1, f_2, f_3\right) &= f_i \quad (i = 1, 2, 3) \\ F_3^3\left(f, e_1^{(3)}, e_2^{(3)}, e_3^{(3)}\right) &= f \\ F_3^3\left(F_3^3\left(f, g_1, g_2, g_3\right), h_1, h_2, h_3\right) &= \\ & F_3^3(f, F_3^3\left(g_1, h_1, h_2, h_3\right), F_3^3\left(g_2, h_1, h_2, h_3\right), F_3^3\left(g_3, h_1, h_2, h_3\right)) \end{aligned}$$

The clone $\mathcal{C}$ is minimal iff $\mathcal{C}^{(3)}$ has no proper nontrivial (i.e. different from $\{e_1, e_2, e_3\}$) subalgebras. The main result of this chapter is the following theorem that describes minimal clones of type (III) with at most four majority operations, i.e. with at most seven ternary operations. The characterization is given up to the isomorphism of the ternary part of the clone (but not up to the isomorphism of the whole clone!).

Theorem 3.6 **[Wa4]** *There is no minimal clone with exactly two or four majority operations. If $\mathcal{C}$ is a minimal clone with one or three majority operations, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_1]^{(3)}$ or $[m_3]^{(3)}$, respectively* (*see Table* 3).

In Section 3.1 we prepare the proof of this theorem by proving a statement about the possible symmetries of majority operations in a minimal clone, and we also examine the simplest case, when there is just one majority operation in the clone. Section 3.2 contains the hard part of the proof: the cases of 2, 3 and 4 majority operations.

3.1 Symmetries of minimal majority functions

For any abstract clone $\mathcal{C}$, the symmetric group S_n acts naturally on $\mathcal{C}^{(n)}$: applying a permutation $\pi \in S_n$ to $f \in \mathcal{C}^{(n)}$ we get

$$f\left(e_{\pi(1)}^{(n)}, e_{\pi(2)}^{(n)}, \ldots, e_{\pi(n)}^{(n)}\right). \tag{3.1}$$

In the case of concrete clones this means that we permute the variables of f, and we will adopt this terminology to the abstract case, even though we cannot speak about variables here. If f is a nontrivial operation, then so are the operations of the form (3.1), hence S_n acts on $\mathcal{C}^{(3)} \setminus \mathcal{I}$, too. Let us denote by $\sigma(f)$ the stabilizer of f, i.e. the group of permutations leaving f invariant.

If f is a majority operation, then $\sigma(f)$ is a subgroup of S_3, therefore it has $1, 2, 3$ or 6 elements. If $\sigma(f) \supseteq A_3$, then we say that f is *cyclically symmetric*, and if $\sigma(f) = S_3$, then we say that f is *totally symmetric.*

If $\mathcal{C}$ is a majority clone with just one majority operation, then the majority rule and the clone axioms completely determine the structure of $\mathcal{C}^{(3)}$, and it is clear that in this case $\mathcal{C}$ is minimal. For example, $[m_1]$ is such a clone, so we have the following theorem.

Theorem 3.1. **[Wa4]** *If $\mathcal{C}$ is a minimal clone with one majority operation, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_1]^{(3)}$.*

If f is the unique majority operation in such a clone, then every nontrivial ternary superposition of f yields f itself. In particular, f is totally symmetric, and satisfies $f(f(x,y,z), y, z) = f(x,y,z)$. It is easy to check that this identity together with the total symmetry ensures that f does not generate any nontrivial ternary operation other than f, hence the clones described in the above theorem are exactly the factor clones of the clone of the variety $\mathcal{M}_1$ defined by the following identities:

$$f(x,y,z) = f(y,z,x) = f(y,x,z) = f(f(x,y,z), y, z), \ f(x,x,y) = x. \tag{3.2}$$

This variety has infinitely many subvarieties, therefore there are infinitely many nonisomorphic minimal clones with just one majority operation. To see this, we will construct a subdirectly irreducible (in fact, simple) algebra $\mathbb{A}_n \in \mathcal{M}_1$ of size n for every $n > 6$. Since $\mathcal{M}_1$ is congruence distributive, $\mathbb{A}_m \notin \mathrm{HSP}(\mathbb{A}_n)$ if $m > n$ by Jónsson's lemma, hence the subvarieties $\mathrm{HSP}(\mathbb{A}_n)$ are all different, and the clones $\mathrm{Clo}\,\mathbb{A}_n$ are pairwise nonisomorphic.

Example 3.2. Let $\mathbb{A}_n = (\{1, 2, \ldots, n\}; f)$, where f is a totally symmetric majority operation defined for $1 \leq a < b < c \leq n$ by

$$f(a, b, c) = \begin{cases} a & \text{if } \left\lceil \frac{a+c}{2} \right\rceil < b < c; \\ b & \text{if } b = \left\lfloor \frac{a+c}{2} \right\rfloor \text{ or } b = \left\lceil \frac{a+c}{2} \right\rceil; \\ c & \text{if } a < b < \left\lfloor \frac{a+c}{2} \right\rfloor. \end{cases}$$

Note that it suffices to define $f(a, b, c)$ for $a < b < c$ since f is a totally symmetric majority function. Let us consider the elements of A_n as points on the real line. If $a < c$, then we could call the points $\left\lfloor \frac{a+c}{2} \right\rfloor$ and $\left\lceil \frac{a+c}{2} \right\rceil$ the midpoints of the segment between a and c. (Segments of even length have one midpoint, while segments of odd length have two midpoints.) If $a < b < c$ and b is a midpoint of the segment between a and c, then $f(a, b, c) = b$, otherwise $f(a, b, c)$ is that endpoint of this segment which is farther from b.

It is easy to check that $\mathbb{A}_n \in \mathcal{M}_1$ (because f is conservative), and we claim that $\mathbb{A}_n$ is simple if $n > 6$. To prove this, let us first observe that if I is a congruence class, then I has the following property: if at least two of a, b, c belong to I, then $f(a, b, c) \in I$. Let us call such subsets ideals of $\mathbb{A}_n$. If I is an ideal and $a, c \in I$, then I contains the midpoints of the segment between a and c. Successively taking midpoints we can reach any point between a and c, therefore this whole segment belongs to I, i.e. ideals are convex.

Let ϑ be a nontrivial congruence of $\mathbb{A}_n$, and let a be the least element of A_n that belongs to a non-singleton block I of ϑ. Since a is the smallest element of I, which is a convex set with at least two elements, we must have $a+1 \in I$. If $a \geq 4$, then $f(1, a, a+1) = 1$, and by the ideal property $f(1, a, a+1) \in I$. Now $2 \in I$ follows by convexity, and then $n = f(1, 2, n) \in I$ (here we need that $n \geq 5$). As both 1 and n belong to I, we have $I = \{1, 2, \ldots, n\}$, i.e. ϑ is the total relation on $\mathbb{A}_n$.

If $a + 1 \leq n - 3$, then a similar argument works: $n = f(a, a+1, n) \in I$, and then $1 = f(1, n-1, n) \in I$, therefore ϑ is the total relation again. The assumption $n > 6$ ensures that at least one of $a \geq 4$ and $a + 1 \leq n - 3$ holds, hence $\mathbb{A}_n$ is simple, as claimed.

From now on $\mathcal{C}$ will denote an arbitrary majority minimal clone. To simplify the notation we will just write 1, 2 and 3 for the first, second and third ternary projections respectively, and numbers greater than 3 will denote nontrivial elements of $\mathcal{C}^{(3)}$. Our next goal is to prove that if all majority functions in $\mathcal{C}$ are cyclically symmetric, then there is only one majority operation in the clone, i.e. $\mathcal{C}^{(3)} \cong [m_1]^{(3)}$.

In preparation, we introduce three binary operations on the ternary part of $\mathcal{C}$.

$$
\begin{aligned}
f * g &= f\left(g\left(1,2,3\right), g\left(2,3,1\right), g\left(3,1,2\right)\right) \\
f \bullet g &= f\left(g\left(1,2,3\right), 2, 3\right) \\
f \circledcirc g &= f\left(1, g\left(1,2,3\right), g\left(1,3,2\right)\right)
\end{aligned}
$$

The proof of the next theorem is similar to the proof of Theorem 2.1 (note that $*$ is the same operation as the one introduced there). Concerning the operation $\bullet$ see also Lemma 4.4 of [HM].

Theorem 3.3. **[Wa4]** *The operations $*$, $\bullet$ and $\circledcirc$ are associative, and if $\mathcal{C}$ is a majority clone, then $\mathcal{C}^{(3)} \setminus \mathcal{I}$ is closed under them. Therefore if $\mathcal{C}^{(3)}$ is finite, then it contains a nontrivial idempotent element for each of these operations.*

Proof. It is easy to check that if f and g are majority operations, then so are $f * g$, $f \bullet g$ and $f \circledcirc g$, hence $\mathcal{C}^{(3)} \setminus \mathcal{I}$ is closed under these three operations. Associativity can be checked by a routine calculation using the three defining axioms of abstract clones. We work out the details for $\circledcirc$, the other two cases are similar. Let us compute $(f \circledcirc g) \circledcirc h$ first:

$$
\begin{gathered}
(f \circledcirc g) \circledcirc h = (f \circledcirc g)\left(1, h\left(1,2,3\right), h\left(1,3,2\right)\right) = \\
f\left(1, g\left(1, h\left(1,2,3\right), h\left(1,3,2\right)\right), g\left(1, h\left(1,3,2\right), h\left(1,2,3\right)\right)\right).
\end{gathered}
$$

For $f \circledcirc (g \circledcirc h)$ we have

$$
\begin{gathered}
f \circledcirc (g \circledcirc h) = f\left(1, (g \circledcirc h)\left(1,2,3\right), (g \circledcirc h)\left(1,3,2\right)\right) = \\
f\left(1, g\left(1, h\left(1,2,3\right), h\left(1,3,2\right)\right)\left(1,2,3\right), g\left(1, h\left(1,2,3\right), h\left(1,3,2\right)\right)\left(1,3,2\right)\right) = \\
f\left(1, g\left(1, h\left(1,2,3\right), h\left(1,3,2\right)\right), g\left(1, h\left(1,3,2\right), h\left(1,2,3\right)\right)\right).
\end{gathered}
$$

The last statement of the theorem follows since every finite semigroup contains an idempotent element. $\square$

Now we are ready to prove the main result of this section. This theorem is an analogue of a theorem of J. Dudek and J. Gałuszka which states that if a binary minimal clone contains finitely many nontrivial binary operations all of which are commutative, then there is just one nontrivial binary operation in the clone [DG].

Theorem 3.4. **[Wa4]** *Let $\mathcal{C}$ be a majority minimal clone with finitely many ternary operations. If every nontrivial ternary operation in $\mathcal{C}$ is cyclically symmetric, then $\mathcal{C}$ contains only one nontrivial ternary operation, hence $\mathcal{C}^{(3)} \cong [m_1]^{(3)}$.*

Proof. Let $\mathcal{C}^{(3)} = \{1, 2, \ldots, n\}$, where $1, 2, 3$ are the ternary projections as before. First let us assume that there is no totally symmetric majority function in $\mathcal{C}$, i.e. $\sigma(f) = A_3$ for all $f \geq 4$. By Theorem 3.3 there is a nontrivial $\circledcirc$-idempotent, say $4 \circledcirc 4 = 4$. Since 4 is not invariant under the transposition (23), the element $4(1, 3, 2)$ is different from 4, thus we may suppose without loss of generality that $4(1, 3, 2) = 5$. We have $4(1, 4, 5) = 4 \circledcirc 4 = 4$, hence $4|_{\langle 145 \rangle} \equiv 4$ because 4 is cyclically symmetric. We can compute $4(1, 5, 4)$ as well, using the associativity of composition:

$$4(1, 5, 4) = 4(1(1, 3, 2), 4(1, 3, 2), 5(1, 3, 2)) = 4(1, 4, 5)(1, 3, 2) = 4(1, 3, 2) = 5.$$

Thus we have $4|_{\langle 154 \rangle} \equiv 5$, therefore 4 preserves $\{1, 4, 5\}$, and its restriction to this set is isomorphic to m_3. However, m_3 generates majority operations that are not cyclically symmetric (see Table 3), and this contradicts our assumption that every nontivial ternary operation of $\mathcal{C}$ is cyclically symmetric. This contradiction shows that $\mathcal{C}$ must contain at least one totally symmetric majority function. If f and g are totally symmetric, then $f \bullet g$ is invariant under the transposition (23):

$$\begin{aligned}(f \bullet g)(1, 3, 2) = f(g(1, 2, 3), 2, 3)(1, 3, 2) = \\ f(g(1, 3, 2), 3, 2) = f(g(1, 2, 3), 2, 3) = f \bullet g.\end{aligned}$$

Since $f \bullet g$ is nontrivial, it is also cyclically symmetric, hence $\sigma(f \bullet g) = S_3$. Thus totally symmetric majority functions form a finite semigroup under $\bullet$, so there is a totally symmetric $f \in \mathcal{C}^{(3)}$ with $f \bullet f = f$. Then f satisfies the identities in (3.2), hence $[f]^{(3)} \cong [m_1]^{(3)}$. By the minimality of $\mathcal{C}$ we have $[f] = \mathcal{C}$, and this proves the theorem. $\square$

Corollary 3.5. [Wa4] *If $\mathcal{C}$ is a majority minimal clone with $2 \leq \left|\mathcal{C}^{(3)}\right| < \aleph_0$, then the action of S_3 on $\mathcal{C}^{(3)} \setminus \mathcal{I}$ has an orbit with at least 3 elements.*

Proof. By the previous theorem there is a nontrivial operation $f \in \mathcal{C}^{(3)}$ which is not cyclically symmetric. Thus $\sigma(f)$ has at most 2 elements, and therefore the size of the orbit of f is $6/|\sigma(f)| \geq 3$. $\square$

3.2 Minimal clones with at most four majority operations

In this section we prove the main result of this chapter, the following characterization of majority minimal clones with at most seven ternary operations.

Theorem 3.6. [Wa4] *There is no minimal clone with exactly two or four majority operations. If $\mathcal{C}$ is a minimal clone with one or three majority operations, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_1]^{(3)}$ or $[m_3]^{(3)}$, respectively (see Table 3).*

Theorem 3.1 describes the minimal clones with one majority operation, and from Corollary 3.5 we see immediately that there is no minimal clone with exactly two majority operations. We will deal with the cases of three and four majority operations in two separate lemmas.

Lemma 3.7. [Wa4] *If $\mathcal{C}$ is a minimal clone with three majority operations, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_2]^{(3)}$.*

Proof. Let $\mathcal{C}$ be a minimal clone with three majority functions, and let $\mathcal{C}^{(3)} = \{1,2,3,4,5,6\}$, where $1,2,3$ are the ternary projections. Considering the orbits of the action of S_3 on $\{4,5,6\}$ we see by Corollary 3.5 that the only possibility is that there is just one orbit, i.e. any two nontrivial ternary operations can be obtained form each other by cyclic permutations of variables. We can suppose that $4\,(2,3,1) = 5$ and $5\,(2,3,1) = 6$ (and then $6\,(2,3,1) = 4$). Any composition of majority operations is again a majority operation, therefore the set $\mathcal{C}^{(3)} \setminus \mathcal{I} = \{4,5,6\}$ is preserved by 4. This implies that every operation in $\mathcal{C}$ preserves $\{4,5,6\}$, since $\mathcal{C} = [4]$. Thus we have a clone homomorphism

$$\varphi : \mathcal{C} \to \mathcal{O}_{\{4,5,6\}},\ f \mapsto f|_{\{4,5,6\}}.$$

We claim that φ is injective on $\{1,2,3,4,5,6\}$. Clearly it suffices to show that $\varphi(4) \neq \varphi(5) \neq \varphi(6) \neq \varphi(4)$. We prove the first unequality, the other two are similar. Let us compute $5\,(4,5,6)$ using the associativity of composition:

$$5\,(4,5,6) = 4\,(2,3,1)\,(4,5,6) = 4\,(5,6,4) =$$
$$4\,(4\,(2,3,1)\,, 5\,(2,3,1)\,, 6\,(2,3,1)) = 4\,(4,5,6)\,(2,3,1)\,.$$

Since $4\,(4,5,6) \in \{4,5,6\}$ and none of these three elements are invariant under the permutation (231), we have $5\,(4,5,6) = 4\,(4,5,6)\,(2,3,1) \neq 4\,(4,5,6)$. Thus $4|_{\{4,5,6\}} \neq 5|_{\{4,5,6\}}$ as claimed. Now we see that $\mathcal{C}^{(3)}$ is isomorphic to its image under φ, which is the ternary part of a minimal clone on a three-element set. Therefore $\mathcal{C}^{(3)} \cong [m_i]^{(3)}$ for some $i \in \{1,2,3\}$. The cardinality of $\mathcal{C}^{(3)}$ is 6, so we must have $i = 2$, and the lemma is proved. $\square$

Remark 3.8. The previous lemma can be formulated in terms of algebras and varieties as follows. Let $\mathcal{M}_2$ be the variety defined by the three-variable identities satisfied by $(\{1,2,3\}\,; m_2)$. If f is a majority operation on a set A, then $[f]$ is a minimal clone with exactly three majority operations iff $(A; f)$ is term equivalent to an element of $\mathcal{M}_2 \setminus \mathcal{M}_1$. Note that no two different subvarieties of $\mathcal{M}_2$ are term equivalent, since for any $\mathbb{A} = (A; f) \in \mathcal{M}_2$ the basic operation f is the only nontrivial ternary function in $\operatorname{Clo} \mathbb{A}$ which is invariant under the transposition (23). This means that in order to show that there are infinitely many nonisomorphic minimal clones with three majority operations, it suffices to verify that the variety $\mathcal{M}_2$ has infinitely many subvarieties that are not contained in $\mathcal{M}_1$. If d_A is the dual discriminator function on a set A with at least three elements, then $(A; d_A(z, y, x)) \in \mathcal{M}_2 \setminus \mathcal{M}_1$, and by Jónsson's lemma we have $(B; d_B(z, y, x)) \notin \mathrm{HSP}(A; d_A(z, y, x))$ if A is finite and $|A| < |B|$. Thus the algebras $(A; d_A(z, y, x))$ with $A = \{1, 2, \ldots, n\}$ and $n \geq 3$ generate pairwise different subvarieties of $\mathcal{M}_2$ that are not contained in $\mathcal{M}_1$.

Lemma 3.9. **[Wa4]** *There is no minimal clone with four majority operations.*

Proof. Let us suppose that $\mathcal{C}$ is a minimal clone with four majority functions, and let $\mathcal{C}^{(3)} = \{1, 2, 3, 4, 5, 6, 7\}$, with $1, 2, 3$ being the ternary projections. Corollary 3.5 shows that there are two orbits under the action of S_3 on $\{4, 5, 6, 7\}$: a three-element and a one-element orbit. Thus one of the four nontrivial operations is totally symmetric, the other three operations have two-element invariance groups, and the latter three functions can be obtained from each other by cyclic permutations of their variables. We may assume without loss of generality that 7 is totally symmetric, and 4, 5 and 6 are invariant under the transpositions (23), (13) and (12) respectively. Then we must have $4(2, 3, 1) = 5$, $5(2, 3, 1) = 6$ and $6(2, 3, 1) = 4$.
Since any composition of majority operations is nontrivial, every operation in $\mathcal{C}$ preserves $\{4, 5, 6, 7\}$. Restricting to this set, we obtain (the ternary part of) a minimal clone on a four-element set. The operation $7(4, 5, 6)$ is easily seen to be totally symmetric: applying a permutation to $7(4, 5, 6)$ will just permute $4, 5$ and 6 in the arguments of 7, and this has no effect on the final value, as 7 is totally symmetric. Since the only totally symmetric operation in $\mathcal{C}^{(3)}$ is 7, we must have $7(4, 5, 6) = 7$. This means that the restriction of 7 to $\{4, 5, 6, 7\}$ is a totally symmetric minimal majority operation that is not conservative. Now Theorem 2.20 implies that $7|_{\{4,5,6,7\}}$ is isomorphic to M_1, so $7(a, b, c) = 7$ for any pairwise distinct $a, b, c \in \{4, 5, 6, 7\}$. Moreover, since M_1 does not generate any majority operation but itself, the operations $4, 5, 6, 7$ coincide with each other on $\{4, 5, 6, 7\}$:

$$f(a, b, c) = 7 \text{ if } f, a, b, c \in \{4, 5, 6, 7\} \text{ and } a, b, c \text{ are pairwise distinct.} \tag{3.3}$$

In particular, we have $6(6, 4, 5) = 7$, and taking into account that 4 and 5 are obtained from 6 by cyclic permutations of variables, this means that $6 * 6 = 7$.
In what follows, we will compute many more compositions until we get a contradiction by constructing a nontrivial ternary operation in $\mathcal{C}$ which is different from $4, 5, 6$ and 7. The operation $7(1, 2, 7)$ is invariant under the transposition (12), hence it is either 6 or 7. The latter is impossible, since $7(1, 2, 7) = 7$ implies that 7 satisfies the identities in (3.2), and then the clone generated by 7 would contain just one nontrivial ternary operation. Thus we have $7(1, 2, 7) = 6$, and by the

total symmetry of 7 it follows that

$$7\,(1,2,7) = 7\,(7,1,2) = 7\,(2,7,1) = 6. \tag{3.4}$$

Let us now consider the values of 6 on $(1,2,7)\,,(2,7,1)\,,(7,1,2)$. We have $6\,(1,2,7) \in \{6,7\}$ since $6\,(1,2,7)$ is invariant under (12). Applying this transposition to $6\,(2,7,1)$ we obtain $6\,(7,1,2)$:

$$6\,(2,7,1)\,(2,1,3) = 6\,(1,7,2) = 6\,(7,1,2)\,.$$

Therefore either both $6\,(2,7,1)$ and $6\,(7,1,2)$ are equal to 6 or 7, or one of them is 4, the other one is 5. The resulting eight possibilities are summarized in the following table.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
6\,(1,2,7) & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 \\
6\,(2,7,1) & 7 & 6 & 4 & 5 & 7 & 6 & 4 & 5 \\
6\,(7,1,2) & 7 & 6 & 5 & 4 & 7 & 6 & 5 & 4 \\
\hline
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{}
\end{array} \tag{3.5}$$

Let us consider any of the eight columns, and let a, b, c be the elements in this column. Then using the fact that $7 = 6 * 6$, we obtain

$$7\,(1,2,7) = 6\,(6\,(1,2,7)\,,6\,(2,7,1)\,,6\,(7,1,2)) = 6\,(a,b,c)\,.$$

For the two columns marked by the arrows this gives $7\,(1,2,7) = 6$ by the majority rule. Similarly, for the first and the fifth column the majority rule yields $7\,(1,2,7) = 7$, and in the remaining four cases we get $7\,(1,2,7) = 7$ again, according to (3.3). However, we already know from (3.4) that $7\,(1,2,7) = 6$, so one of the two possibilities indicated by the arrows takes place. In both cases we have

$$6\,(2,7,1) = 6. \tag{3.6}$$

Now we go on to collect some information about the function 7. For the reader's convenience, we put the number of the equation being used over the equality sign in the following calculations. First of all, using (3.4) and (3.6) we obtain

$$7\,(6,2,7) \overset{(3.4)}{=} 7\,(7,1,2)\,(2,7,1) \overset{(3.4)}{=} 6\,(2,7,1) \overset{(3.6)}{=} 6.$$

Permuting variables we get

$$7\,(4,3,7) = 7\,(6,2,7)\,(2,3,1) = 6\,(2,3,1) = 4; \tag{3.7a}$$
$$7\,(5,3,7) = 7\,(6,2,7)\,(1,3,2) = 6\,(1,3,2) = 5. \tag{3.7b}$$

We already know from (3.3) that $7\,(4,5,7) = 7$, and let us suppose for a moment that $7\,(4,5,3) = 7$. Then (3.7) shows that 7 preserves $\{3,4,5,7\}$, and its restriction to this four-element set is a totally symmetric nonconservative minimal majority function. Therefore it is isomorphic to M_1 by Theorem 2.20. However, this is clearly not the case. This contradiction shows that $7\,(4,5,3) \neq 7$. Let us observe that $7\,(4,5,3)\,(2,1,3) = 7\,(5,4,3) = 7\,(4,5,3)$, i.e. $7\,(4,5,3)$ is invariant under the transposition (12). Since 6 and 7 are the only nontrivial functions in our clone which are invariant under (12), we must have

$$7\,(4,5,3) = 6. \tag{3.8}$$

Next we calculate the value of $6\,(4,5,3)$:

$$6\,(4,5,3) \overset{(3.4)}{=} 7\,(1,2,7)\,(4,5,3) \overset{(3.8)}{=} 7\,(4,5,6) \overset{(3.3)}{=} 7. \tag{3.9}$$

Note that $6\,(3,4,5)\,(2,1,3) = 6\,(3,5,4) = 6\,(5,3,4)$, hence similarly to the previous table, we can list the possible behaviours of 6 on $\{(4,5,3)\,,(5,3,4)\,,(3,4,5)\}$.

$$\begin{array}{|c|c|c|c|c|} \hline 6\,(4,5,3) & 7 & 7 & 7 & 7 \\ 6\,(5,3,4) & 7 & 6 & 5 & 4 \\ 6\,(3,4,5) & 7 & 6 & 4 & 5 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \end{array} \tag{3.10}$$

We can read $7\,(4,5,3)$ from this table in the same way as we read $7\,(1,2,7)$ from (3.5). We see that $7\,(4,5,3) = 7$ in three of the four cases. However, we already know that $7\,(4,5,3) \overset{(3.8)}{=} 6$, so the only possibility is the one marked by the arrow. Finally, to reach the desired contradiction, let us consider $6\,(2,3,6)$. Denoting this composition by f, we show that $f\,(4,5,3) = 5$:

$$f\,(4,5,3) = 6\,(2,3,6)\,(4,5,3) \overset{(3.9)}{=} 6\,(5,3,7) \overset{(3.4)}{=} 7\,(1,2,7)\,(5,3,7) \overset{(3.7b)}{=} 7\,(5,3,5) = 5.$$

The operation f is nontrivial, but it does not coincide with any of 4, 5, 6 or 7, because the value of these functions on $(4,5,3)$ is different from 5. Indeed, we have

$$\begin{aligned} 4\,(4,5,3) &= 6\,(5,3,4) \overset{(3.10)}{=} 6; \\ 5\,(4,5,3) &= 6\,(3,4,5) \overset{(3.10)}{=} 6; \\ 6\,(4,5,3) &\overset{(3.9)}{=} 7; \\ 7\,(4,5,3) &\overset{(3.8)}{=} 6. \end{aligned}$$

Thus we have more than four majority operations in our clone, and this contradiction completes the proof. $\square$

Chapter 4

Minimal clones with weakly abelian representations

Our goal in this chapter is to generalize Theorem 1.13 using a weaker term condition. Let us first recall the definition of abelianness together with three other term conditions (cf. [KK]). For an algebra $\mathbb{A}$ let $\mathcal{M}(\mathbb{A})$ denote the set of 2×2 matrices of the form $\begin{pmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{pmatrix}$ where t is a polynomial of $\mathbb{A}$ of arity $n+m$ and $\mathbf{a}, \mathbf{b} \in A^n$, $\mathbf{c}, \mathbf{d} \in A^m$. We say that the algebra $\mathbb{A}$ is

(1) *weakly abelian*, if $\begin{pmatrix} u & u \\ u & v \end{pmatrix} \in \mathcal{M}(\mathbb{A})$ implies $u = v$;

(2) *abelian*, if $\begin{pmatrix} u & u \\ v & w \end{pmatrix} \in \mathcal{M}(\mathbb{A})$ implies $v = w$;

(3) *rectangular*, if $\begin{pmatrix} u & v \\ w & u \end{pmatrix} \in \mathcal{M}(\mathbb{A})$ implies $u = v = w$;

(4) *strongly abelian*, if it is both abelian and rectangular.

All of these properties are inherited by subalgebras and direct products, but not by homomorphic images. If $\mathbb{A}$ is a groupoid, and we apply (1) to $t(x, y) = xy$, then we get that whenever in the multiplication table of $\mathbb{A}$ we see a configuration like this:

$$\begin{array}{c|ccccccc} \cdot & \cdots & c & \cdots & d & \cdots \\ \hline \vdots & & \vdots & & \vdots & \\ a & \cdots & u & \cdots & u & \cdots \\ \vdots & & \vdots & & \vdots & \\ b & \cdots & u & \cdots & v & \cdots \\ \vdots & & \vdots & & \vdots & \end{array}$$

then we must have $u = v$. Of course, this is just a necessary condition for $\mathbb{A}$ to be weakly abelian.

The main result of this chapter is the following characterization of minimal clones with weakly abelian representations.

Theorem 4.16 **[Wa2]** *If a minimal clone has a nontrivial weakly abelian representation, then it also has a nontrivial abelian representation. Therefore such a clone must be a unary clone, the clone of an affine space, a rectangular band or a p-cyclic groupoid for some prime p.*

It was proved in Theorem 3.1 of [Kea] that minimal clones of type (III) and (V) do not have nontrivial abelian representations, and the proof actually shows that they do not have nontrivial weakly abelian representations either. Every representation of a minimal clone of type (I) or (IV) is clearly abelian, therefore we only need to consider weakly abelian groupoids with a minimal clone. In Section 4.1 we discuss the relationship between weak abelianness and distributivity in idempotent groupoids; we describe distributive groupoids with a minimal clone, and we prove that if a weakly abelian groupoid has a minimal clone, then at least one of the distributive laws hold. Section 4.2 finishes the proof of Theorem 4.16 by characterizing weakly abelian groupoids satisfying one-sided distributivity and having a minimal clone. Section 4.3 contains some corollaries. It will turn out that if a minimal clone has a (weakly) abelian representation, then every representation is weakly abelian, but not necessarily abelian. From Theorem 4.16 we will easily obtain the list of minimal clones with rectangular and strongly abelian representations, and we will see that if a minimal clone has a nontrivial rectangular representation, then it also has a nontrivial strongly abelian representation; moreover, all representations are strongly abelian.

4.1 Weak abelianness and distributivity

In the theory of quasigroups a different notion of 'weak abelianness' is defined by the identities

$$(xx)(yz) = (xy)(xz), \qquad (yz)(xx) = (yx)(zx), \tag{4.1}$$

and a groupoid is called 'abelian' (or medial, or entropic) if $(xy)(zu) = (xz)(yu)$ holds (see [Kep]). To avoid confusion with the universal algebraic definitions, we will use the word *entropic* in the latter case. Minimal clones are always idempotent, and in this case the identities (4.1) are equivalent to the *distributive laws*:

Left distributivity: $x(yz) = (xy)(xz)$,
Right distributivity: $(yz)x = (yx)(zx)$.

Any idempotent abelian groupoid is entropic [Kea], and one might expect that idempotent weakly abelian groupoids are distributive. We do not know if this is true or not, but for our present purposes the weaker properties stated in the next two lemmas are sufficient.

Lemma 4.1. **[Wa2]** *If $\mathbb{A}$ is an idempotent weakly abelian groupoid and $u, v_1, v_2 \in A$, then $uv_1 = uv_2 = w$ implies $u(v_1v_2) = w$, i.e. $\{v : uv = w\}$ is a subuniverse for any given $u, w \in A$.*

Proof. Applying the definition of weak abelianness with $\mathbf{a} = (u, v_1, u), \mathbf{b} = (u, u, v_1)$, $\mathbf{c} = v_1, \mathbf{d} = v_2$ for $t(x_1, x_2, x_3, x_4) = (x_1x_2)(x_3x_4)$ we get

$$\begin{pmatrix} (uv_1)(uv_1) & (uv_1)(uv_2) \\ (uu)(v_1v_1) & (uu)(v_1v_2) \end{pmatrix} = \begin{pmatrix} ww & ww \\ uv_1 & u(v_1v_2) \end{pmatrix} = \begin{pmatrix} w & w \\ w & u(v_1v_2) \end{pmatrix} \in \mathcal{M}(\mathbb{A}),$$

hence $u(v_1v_2) = w$. □

Lemma 4.2. **[Wa2]** *Every idempotent weakly abelian groupoid satisfies the following identities:*

(i) $(xy)(xz) = (x(yz))((xy)(xz))$;

(ii) $(yx)(zx) = ((yx)(zx))((yz)x)$;

(iii) $(xy)x = x(yx)$.

Proof. Let $\mathbb{A}$ be an idempotent weakly abelian groupoid. To prove (i), we will use the 8-ary term $((x_1x_2)(x_3x_4))((x_5x_6)(x_7x_8))$; the underlined letters show the entries occupied by $\mathbf{c}$ and $\mathbf{d}$ in the definition of weak abelianness. We have

$$\begin{pmatrix} ((xy)(x\underline{y}))((x\underline{x})(\underline{z}z)) & ((xy)(x\underline{z}))((x\underline{y})(\underline{x}z)) \\ ((xx)(y\underline{y}))((x\underline{x})(\underline{z}z)) & ((xx)(y\underline{z}))((x\underline{y})(\underline{x}z)) \end{pmatrix}$$
$$= \begin{pmatrix} (xy)(xz) & (xy)(xz) \\ (xy)(xz) & (x(yz))(xy)(xz) \end{pmatrix} \in \mathcal{M}(\mathbb{A}),$$

therefore the equality in (i) holds. Doing the same with the dual of $\mathbb{A}$, which is of course also weakly abelian, we obtain the second identity. We could derive the third identity in a similar manner, but it is easier to deduce it from the previous ones. If we put $z = x$ in (i) we get $(xy)x = (x(yx))((xy)x)$; replacing y with x and z with y in (ii) yields $x(yx) = (x(yx))((xy)x)$; comparing them gives (iii). □

In light of the last identity we will sometimes omit the parentheses in products of the form xyx. To make the connection between distributivity and weak abelianness more explicit, we will define a binary relation $\sim$ on our groupoid by

$a \sim b$ iff $ab = a$. Identity (ii) says that $\mathbb{A}$ is right distributive 'modulo $\sim$'. This does not make perfect sense yet, since $\sim$ may not be a congruence, maybe not even an equivalence relation. Our strategy will be to reduce the problem to the case when $\sim$ is a congruence relation. As a preparation, we first show that assuming that the clone of $\mathbb{A}$ is minimal, we can conclude that $\mathbb{A}$ satisfies at least one-sided distributivity.

Lemma 4.3. **[Wa2]** *A weakly abelian groupoid with a minimal clone must satisfy at least one of the distributive laws.*

Proof. Suppose that $\mathbb{A}$ is a weakly abelian groupoid with a minimal clone, and $\mathbb{A}$ is neither left nor right distributive. First we will show that there is a two-element left zero semigroup in $\mathrm{HSP}(\mathbb{A})$. Since $\mathbb{A}$ is not right distributive, we can find elements x, y, z such that $b = (yz)x \neq (yx)(zx) = a$. The second identity of Lemma 4.2 shows that $ab = a$. If $ba = b$, then $\{a, b\}$ is a two-element left zero subsemigroup of $\mathbb{A}$. If $ba \neq b$, then let c denote the product ba, which is different from a by the weak abelian property. We have $ab = aa = a$, so Lemma 4.1 yields that $a = a(ba) = ac$. With the help of identity (iii) of Lemma 4.2 we can compute $cb = (ba)b = b(ab) = ba = c$. Thus we have the following part in the multiplication table of $\mathbb{A}$.

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & a & a \\ b & c & b & \\ c & & c & c \end{array}$$

If $bc = b$, then again we have a two-element left zero subsemigroup, namely $\{b, c\}$. Suppose therefore that $bc \neq b$. Then $x(xy)$ is a nontrivial operation, since $a(ab) = aa = a \neq b$ and $b(ba) = bc \neq b$. However, the operation $x(xy)$ is trivial on the set $\{a, c\}$. The only entry which we need to verify is $c(ca) = c$. We can get this equality by simply applying the definition of weak abelianness on the following matrix:

$$\begin{pmatrix} c(\underline{b}b) & c(\underline{c}b) \\ c(\underline{b}a) & c(\underline{c}a) \end{pmatrix} = \begin{pmatrix} c & c \\ c & c(ca) \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

Therefore any operation in the clone generated by $x(xy)$ is a projection on $\{a, c\}$, and the original multiplication must be in this clone since it was supposed to generate a minimal clone. Thus we have $ca = c$, that is, $\{a, c\}$ is a two-element left zero subsemigroup. Passing from $\mathbb{A}$ to its dual, which is not left or right distributive either, we see from the fact proved in the preceding paragraph that $\mathbb{A}$ also has a two-element right zero subsemigroup. The product of these two is a nontrivial rectangular band in $\mathrm{HSP}(\mathbb{A})$, therefore Lemma 1.5 implies that $\mathbb{A}$ itself is a rectangular band. This is a contradiction, since rectangular bands are distributive. $\square$

With the help of Lemma 4.3 we will be able to handle all cases where $\sim$ is not a congruence relation, and finally we will arrive at the quotient groupoid $\mathbb{A}/\sim$, which will turn out to be distributive. This will be a rather lengthy argument, so we postpone it to the next section. Here we give the characterization of distributive groupoids with a minimal clone, which we will need to analyse $\mathbb{A}/\sim$.

It was shown in [KN] that every distributive groupoid is trimedial, i.e. any subgroupoid generated by at most three elements is entropic. The next theorem shows that the distributive and entropic properties are equivalent for groupoids with a minimal clone, hence we get the same list of groupoids as in Theorem 1.12.

Theorem 4.4. [Wa2] *If $\mathbb{A}$ is a distributive groupoid with a minimal clone, then the entropic law holds in $\mathbb{A}$, therefore $\mathbb{A}$ or its dual is an affine space, a rectangular band, a left normal band, a right semilattice or a p-cyclic groupoid for some prime p.*

Proof. We know that all three-generated subgroupoids of $\mathbb{A}$ are entropic. If they are all trivial, then there must be a left and a right zero semigroup among them (otherwise the clone of $\mathbb{A}$ would be trivial), and the product of these gives a nontrivial rectangular band in HSP($\mathbb{A}$). Applying Lemma 1.5, we get that $\mathbb{A}$ is a rectangular band. If there is a nontrivial 3-generated subalgebra which is an affine space, a rectangular band, or (the dual of) a p-cyclic groupoid, then again by Lemma 1.5 we have that $\mathbb{A}$ (or its dual) belongs to one of these varieties. Hence in all these cases $\mathbb{A}$ is entropic.
So we can assume that every three-generated subgroupoid of $\mathbb{A}$ is a left or right semilattice or a normal band. If there is a nontrivial right semilattice among them, then the term $x(xy)$ is the first projection on this subalgebra, hence by the minimality of the clone we have $\mathbb{A} \models x(xy) = x$. This equation does not hold in a left semilattice or in a normal band, except for a left zero semigroup (which is a right semilattice). Thus we have that every 3-generated subalgebra is a right semilattice. This means that all identities involving at most three variables which hold in the variety of right semilattices also hold in $\mathbb{A}$. Since right semilattices are axiomatizable by three-variable identities, we conclude that $\mathbb{A}$ itself is a right semilattice.
The case of left semilattices is similar, so finally we can suppose that we have only normal bands as 3-generated subalgebras, i.e. that $\mathbb{A}$ satisfies all 3-variable identities that hold for normal bands. Associativity is such an identity, so our groupoid is a distributive semigroup, hence entropic: $xyzu = xyzyu = xzyu$ (cf. [KN], Proposition 2.3). $\square$

Finally let us see which of the varieties mentioned in Theorem 1.12 contain nontrivial weakly abelian algebras.

Theorem 4.5. **[Wa2]** *If $\mathbb{A}$ is a weakly abelian entropic groupoid with a minimal clone, then $\mathbb{A}$ or its dual is a rectangular band, an affine space or a p-cyclic groupoid for some prime p.*

Proof. By Theorem 1.12, we only need to show that $\mathbb{A}$ cannot be a left or right normal band, or left or right semilattice. A nontrivial semilattice is clearly not weakly abelian. In a nontrivial right normal band one can find elements a, b such that $b \neq ab$. It is easy to check that $\{b, ab\}$ is a two-element subsemilattice, contradicting weak abelianness. Similarly, a nontrivial left normal band cannot be weakly abelian either. Finally, let us suppose that $\mathbb{A}$ is a right semilattice (the case of a left semilattice is similar). Considering the matrix

$$\begin{pmatrix} (x\underline{y})(yy) & (x\underline{x})(yy) \\ (x\underline{y})(xy) & (x\underline{x})(xy) \end{pmatrix} = \begin{pmatrix} xy & xy \\ xy & x \end{pmatrix} \in \mathcal{M}(\mathbb{A})$$

we see that $xy = x$ holds for all $x, y \in A$, and this contradicts the assumption that $\mathbb{A}$ has a minimal clone. □

4.2 Left distributive weakly abelian groupoids with minimal clones

Throughout this section $\mathbb{A}$ will denote a weakly abelian groupoid with a minimal clone. Lemma 4.3 shows that such a groupoid satisfies at least one of the distributive laws, so we will suppose that $\mathbb{A}$ is left distributive. We define a binary relation $\sim$ on $\mathbb{A}$ by $a \sim b$ iff $ab = a$. Clearly, this relation is reflexive. First we prove that if $\sim$ is not a congruence, then $\mathbb{A}$ is a p-cyclic groupoid for some prime p. (Note that in the first claim we do not use left distributivity.)

Claim 4.6. *If $\sim$ is not symmetric, then $\mathbb{A} \models x(xy) = x$.*

Proof. Suppose that there are elements $a, b \in A$ such that $a \sim b$ but $b \not\sim a$, that is, $ab = a$ and $ba = c \neq b$. This situation is the same as in the proof of Lemma 4.3, and we will proceed similarly, but this time we go farther. Again, we have $c \neq a$ by the weak abelian property. Let $\mathbb{S}$ be the subgroupoid of $\mathbb{A}$ generated by a and b. According to Lemma 4.1, the set $\{x : ax = a\}$ is a subuniverse of $\mathbb{A}$, and it contains a and b. Therefore it contains S, which implies that a is a left zero element in this subgroupoid. Moreover, $xy = a$ implies $x = a$ for every $x, y \in S$. This can be seen in the multiplication table of $\mathbb{S}$ by weak abelianness.

$$\begin{array}{c|ccccc} & a & \cdots & x & \cdots & y \\ \hline a & a & \cdots & a & \cdots & a \\ \vdots & \vdots & & \vdots & & \vdots \\ x & * & \cdots & x & \cdots & a \end{array}$$

(Note that we have $xx = x$ by idempotence, and $*$ indicates xa; its value is irrelevant.)
Next we show that c is almost a left zero element in $\mathbb{S}$; more precisely, $cz = c$ for all $z \in S \setminus \{a\}$. Since z is in the subgroupoid generated by a and b, there is a binary term t such that $t(a, b) = z$. We prove $cz = c$ by induction on the length of t. If this length is zero, then either $t(x, y) = x$ or $t(x, y) = y$. The former is impossible because $z \neq a$. In the latter case we have $cb = (ba)b = b(ab) = ba = c$. Now for the induction step suppose that $z = t(a, b) = uv$ with $u = t_1(a, b), v = t_2(a, b)$. Again, $u \neq a$ follows from $z \neq a$, and therefore $cu = c$ by the induction hypothesis. If v is also different from a, then $cv = c$, so $cz = c(uv) = c$ by Lemma 4.1. If $v = a$, then we have to prove $c(ua) = c$. Let us consider the matrix

$$\begin{pmatrix} c(b\underline{b}) & c(b\underline{a}) \\ c(u\underline{b}) & c(u\underline{a}) \end{pmatrix} = \begin{pmatrix} cb & cc \\ c(ub) & c(ua) \end{pmatrix} = \begin{pmatrix} c & c \\ c(ub) & c(ua) \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

We know that $cu = cb = c$, therefore $c(ub) = c$ as before. Therefore our matrix is of the form $\left(\begin{smallmatrix} c & c \\ c & c(ua) \end{smallmatrix}\right)$, hence $cz = c(ua) = c$ by weak abelianness.
What we just proved means that in the multiplication table of the subgroupoid $\mathbb{S}$, the row of c is constant c except for ca, which may be different. In the same way as we proved that $xy = a$ implies $x = a$, we can show that $xy = c$ implies $x = c$ or $y = a$, that is, c can appear only in its own row and in the column of a.
The knowledge we gathered about the multiplication table is enough to see that the operation $x(xy)$ preserves $S \setminus \{c\}$. Indeed, if $x(xy) = c$ for some $x, y \in S$, then either $x = c$ or $xy = a$. The latter is impossible since it would force $x = a$, but then $x(xy) = a \neq c$. However, the original multiplication does not preserve this set, because $ab = c$. Therefore, by the minimality of the clone, $x(xy)$ must be a projection. Since $a(ab) = a \neq b$, it can only be the first projection, i.e. the identity $x(xy) = x$ holds in $\mathbb{A}$. □

Claim 4.7. *If $\sim$ is symmetric but not transitive, then $\mathbb{A} \models x(xy) = x$.*

Proof. Suppose that there are elements $a, b, c \in A$ such that $a \sim b \sim c$ but $a \not\sim c$. Then a, b, c must be pairwise different, because $\sim$ is reflexive by the idempotence of $\mathbb{A}$. A part of the multiplication table looks like this:

	a	b	c
a	a	a	
b	b	b	b
c		c	c

It is easy to check that we have the same in the multiplication table of $x(xy)$. But for this operation we can compute the missing two entries, too, with the help of the left distributive identity:

$$a(ac) = (ab)(ac) = a(bc) = ab = a,$$
$$c(ca) = (cb)(ca) = c(ba) = cb = c.$$

Thus we see that $x(xy)$ is the first projection on the set $\{a, b, c\}$, but the original operation xy is not, because $a \not\sim c$ implies $ac \neq a$. Therefore, by the minimality of the clone of $\mathbb{A}$, $x(xy)$ must be a trivial operation, hence $\mathbb{A}$ satisfies $x(xy) = x$. $\square$

To finish the investigation of the cases where $\sim$ is not an equivalence relation, we will show that a weakly abelian groupoid with a minimal clone satisfying $x(xy) = x$ must be a p-cyclic groupoid. This will be the consequence of the following lemma, where we do not assume weak abelianness.

Lemma 4.8. [Wa2] *If a groupoid has a minimal clone and satisfies the identity $x(yz) = xy$, then it belongs to the variety $\mathcal{D} \cap \mathcal{A}$ or $\mathcal{C}_p$ for some prime p.*

Proof. Suppose that t_1, t_2 are two terms, and the leftmost variable of t_2 is x. Then it can be shown easily by induction on the length of t_2 that the identity $t_1 t_2 = t_1 x$ holds in our groupoid. This means that any term t can be reduced to a left-associated product: $t = \overleftarrow{x \cdot y_1 \cdot \ldots \cdot y_n}$. Let us now compute what happens if we multiply a term by its leftmost variable: $tx = t\underline{t} = t$ because the leftmost variable of the underlined t is also x.
Thus we have the same situation as in Claim 3.9 of [KSz], except that the order of the variables $y_1, \ldots, y_n$ is not irrelevant. However, when we compute binary terms, we do not have to permute them, so every binary term is of the form xy^k, and we can proceed as in [KSz] to show that either $(xy)y = xy$ or $xy^p = x$ holds for some prime number p.
In the first case we are done, so let us suppose that the latter holds. One can check that the term $t(x, y, z) = (((xy^{p-1})z)y)z^{p-1}$ satisfies the identities $t(x, x, z) = t(x, y, x) = t(x, y, y) = x$, i.e., it is a first semiprojection. Therefore t does not generate any nontrivial binary operation, so it must be trivial: $t(x, y, z) = x$. Substituting xy for x in this equality and multiplying both sides from the right by z we get the identity $t(xy, y, z)z = (xy)z$. Computing the left hand side we

obtain the identity $(xz)y = (xy)z$. Thus all the defining identities of the variety of p-cyclic groupoids hold in our groupoid. $\square$

Lemma 4.9. **[Wa2]** *If a weakly abelian groupoid has a minimal clone and satisfies the identity $x(xy) = x$, then it is a p-cyclic groupoid for some prime p.*

Proof. We show that weak abelianness and the identity $x(xy) = x$ imply the stronger identity $x(yz) = xy$. Let $t = t(x, y, z) = x(yz)$, and compute the following matrix:

$$\begin{pmatrix} t(t\underline{z}) & t(t\underline{y}) \\ x(y\underline{z}) & x(y\underline{y}) \end{pmatrix} = \begin{pmatrix} t & t \\ t & xy \end{pmatrix}.$$

Thus we have $x(yz) = xy$ by weak abelianness, and we can apply the previous lemma. The only thing we need to show is that the identity $(xy)y = xy$ cannot hold. We can proceed the same way as we did at the end of the proof of Theorem 4.5 to see that $(xy)y = xy$ would imply $xy = x$. $\square$

So far we have proved that if $\sim$ is not an equivalence relation, then $\mathbb{A}$ is a p-cyclic groupoid. From now on we will assume that $\sim$ is an equivalence relation, and we will force it to be a congruence of $\mathbb{A}$. Using the left distributive identity we can show that $\sim$ is not very far from being a congruence.

Claim 4.10. *For any $a, b, c \in A$, if $a \sim b$ then the following relations are true:*

(i) $ca \sim cb$,

(ii) $(ac)(bc) \sim ac$.

Proof. To prove (i) we simply apply the left distributive law: $(ca)(cb) = c(ab) = ca$. For (ii) let us substitute $x = c, y = a, z = b$ in the identity $(yx)(zx) = ((yx)(zx))((yz)x)$, which holds in $\mathbb{A}$ by Lemma 4.2. Then we get $(ac)(bc) = ((ac)(bc))((ab)c) = ((ac)(bc))(ac)$ which is just what we had to prove. $\square$

It would be nice if we had $ac \sim bc$ in (ii), because then $\sim$ would be a congruence. With the next claim we finish the investigation of the case where $\sim$ is not a congruence.

Claim 4.11. *If $\sim$ is not a congruence relation, then $\mathbb{A}$ is a p-cyclic groupoid.*

Proof. We prove first that for any $a, b, c \in A$, if $a \sim b$ then the subalgebra generated by ac and bc satisfies the identity $x(xy) = x$. The second part of the previous claim shows that $uv \sim u$ holds for $u, v \in S = \{ac, bc\}$. Next we show that this property is inherited when we pass from S to the subgroupoid generated by S. This can be done using the following two rules:

$$(uw \sim u \text{ and } uv \sim u) \Rightarrow (uv)w \sim uv,$$
$$(wu \sim w \text{ and } wv \sim w) \Rightarrow w(uv) \sim w.$$

To check the first one, we calculate $u((uv)w) = (u(uv))(uw) = u(uw) = u$, which shows that $u \sim (uv)w$. We have assumed $u \sim uv$, therefore by transitivity and symmetry $(uv)w \sim uv$ follows. The second one is easier: $w(w(uv)) = w((wu)(wv)) = (w(wu))(w(wv)) = ww = w$.
With these rules one can show by term induction that $uv \sim u$ for all u, v in the subgroupoid generated by S. Hence this subgroupoid satisfies the identity $x(xy) = x$. If $\sim$ is not a congruence, then we can find elements a, b, c such that $a \sim b$ but $ac \not\sim bc$, that is, $(ac)(bc) \neq ac$. If $(ac)(bc) = bc$, then by the second part of Claim 4.10 we would have $bc \sim ac$, which is impossible since $ac \not\sim bc$. Thus the subalgebra generated by $\{ac, bc\}$ is not trivial. Then it has a minimal clone; it is weakly abelian, and satisfies $x(xy) = x$, therefore by Lemma 4.9 it is a nontrivial p-cyclic groupoid in HSP($\mathbb{A}$). With the help of Lemma 1.5 we conclude that HSP($\mathbb{A}$) is the variety of p-cyclic groupoids. □

Let us summarize what we have proved so far in this section.

Theorem 4.12. **[Wa2]** *If $\mathbb{A}$ is a weakly abelian left distributive groupoid with a minimal clone such that the relation $\sim$ defined by $a \sim b \Leftrightarrow ab = a$ is not a congruence, then $\mathbb{A}$ is a p-cyclic groupoid for some prime p.*

So finally we can suppose that $\mathbb{A}$ is a left distributive weakly abelian groupoid with a minimal clone, and $\sim$ is a congruence of $\mathbb{A}$. The corresponding factor groupoid $\mathbb{A}/\sim$ is distributive (right distributivity holds because $\mathbb{A}$ satisfies identity (ii) of Lemma 4.2). Furthermore, $\mathbb{A}/\sim$ has a minimal or trivial clone. Therefore it is entropic by Theorem 4.4, and it must have at least two elements, since $\mathbb{A}$ is not a left zero semigroup. Using the list of entropic groupoids with a minimal clone, we will prove that $\mathbb{A}$ is also entropic. The key observation is that by the definition of $\sim$ we have for any terms t_1, t_2

$$\mathbb{A}/\sim \models t_1 = t_2 \iff \mathbb{A} \models t_1 t_2 = t_1.$$

Claim 4.13. *If $\mathbb{A}/\sim$ has a two-element left or right zero subsemigroup, then $\mathbb{A}$ is entropic. It is impossible to have a two-element semilattice among the subgroupoids of $\mathbb{A}/\sim$.*

Proof. First let us suppose that $X, Y \in A/\sim$ form a left zero semigroup. Then for any $x, y \in X \cup Y$ we have $xy \sim x$. Therefore $x(xy) = x$ holds in $X \cup Y$, which is a nontrivial subgroupoid of $\mathbb{A}$, since X and Y are two different congruence classes. By Lemma 4.9 this subgroupoid must be p-cyclic, and by the minimality of the clone of $\mathrm{HSP}(\mathbb{A})$, Lemma 1.5 implies that $\mathbb{A}$ itself must also be a p-cyclic groupoid. Now suppose that $X, Y \in A/{\sim}$ form a right zero semigroup. Again, $X \cup Y$ is a subgroupoid of $\mathbb{A}$, and $t_1 t_2 = t_1$ holds in this subalgebra whenever the rightmost variables of t_1 and t_2 are the same (i.e., when $t_1 = t_2$ holds in right zero semigroups). Using this fact and the weak abelian property, we can compute $(xy)\,z$ for $x, y, z \in X \cup Y$ as follows:

$$\begin{pmatrix} ((xy)\underline{y})z & ((xy)\underline{z})z \\ ((xx)\underline{y})z & ((xx)\underline{z})z \end{pmatrix} = \begin{pmatrix} (xy)z & (xy)z \\ (xy)z & xz \end{pmatrix} \in \mathcal{M}(\mathbb{A}),$$

therefore the identity $(xy)z = xz$ holds in $X \cup Y$. Similarly, $X \cup Y \models x(yz) = xz$ can be shown by considering the following matrix:

$$\begin{pmatrix} (xz)(\underline{z}z) & (xz)(\underline{y}z) \\ (xx)(\underline{z}z) & (xx)(\underline{y}z) \end{pmatrix} = \begin{pmatrix} xz & xz \\ xz & x(yz) \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

Thus $X \cup Y$ is a rectangular band, and if it is nontrivial, then $\mathbb{A}$ is also a rectangular band by Lemma 1.5, so we are done. If $X \cup Y$ is trivial, then X and Y must be singletons, because X and Y are left zero subsemigroups. Therefore $X \cup Y$ is a right zero subsemigroup in $\mathbb{A}$. Forming the direct product of this with any non-singleton congruence class we get a nontrivial rectangular band in $\mathrm{HSP}(\mathbb{A})$, so $\mathbb{A}$ is also a rectangular band by Lemma 1.5. If all the $\sim$-blocks of A are singletons, then $\mathbb{A} \cong \mathbb{A}/{\sim}$ is distributive, hence entropic by Theorem 4.4.
Finally, let us suppose that $X, Y \in A/{\sim}$ form a semilattice. Then $X \cup Y$ satisfies all equations of the form $t_1 t_2 = t_1$ where $t_1 = t_2$ is valid in every semilattice. Combining this with identity (iii) of Lemma 4.2 allows us to conclude that the identities

$$\begin{aligned} (xy)y &= ((xy)y)(xy) = (xy)(y(xy)) = xy, \\ (xy)x &= ((xy)x)(xy) = (xy)(x(xy)) = xy \end{aligned}$$

hold in $X \cup Y$. Using these identities we can compute the following matrix for $x, y \in X \cup Y$:

$$\begin{pmatrix} (xy)\underline{y} & (xy)\underline{x} \\ (xx)\underline{y} & (xx)\underline{x} \end{pmatrix} = \begin{pmatrix} xy & xy \\ xy & x \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

Thus $X \cup Y$ is a left zero semigroup, contradicting the fact that X and Y are two different congruence classes. $\square$

Theorem 4.14. **[Wa2]** *If $\mathbb{A}$ is a weakly abelian left distributive groupoid with a minimal clone such that the relation $\sim$ defined by $a \sim b \Leftrightarrow ab = a$ is a congruence, then $\mathbb{A}$ is entropic.*

Proof. There are at least two $\sim$-classes, since otherwise $\mathbb{A}$ would be a left zero semigroup. So $\mathbb{A}/\sim$ has at least two elements, and if it is trivial, then we can apply the previous claim. If this is not the case, then $\mathbb{A}/\sim$ must belong to one of the varieties which have entropic minimal clones. In the case of affine spaces, rectangular bands and p-cyclic groupoids Lemma 1.5 shows that $\mathbb{A}$ also belongs to one of these varieties. As we have seen in the proof of Theorem 4.5, a nontrivial left or right normal band always contains a two-element subsemilattice, but Claim 4.13 shows that this is impossible for $\mathbb{A}/\sim$.
Finally, let us assume that $\mathbb{A}/\sim$ is a nontrivial right semilattice. Then it contains elements a, b such that $a \neq ab$. Using the defining identities of the variety of right semilattices, one can check that a and ab form a two-element left zero subsemigroup in $\mathbb{A}/\sim$, so we can apply Claim 4.13 again. Similarly, a nontrivial left semilattice must contain a two-element right zero subsemigroup, so Claim 4.13 applies in this case, too. $\square$

Putting together Theorems 4.12 and 4.14 with Theorem 4.5 we get the main result of this section.

Theorem 4.15. [**Wa2**] *A left distributive weakly abelian groupoid with a minimal clone is either a rectangular band, an affine space or* (*the dual of*) *a p-cyclic groupoid for some prime p.*

4.3 Minimal clones with term conditions

We have seen that only minimal clones of types (I), (II) and (IV) can have nontrivial weakly abelian representations, and in case of types (I) and (IV) all representations are abelian. A weakly abelian groupoid with a minimal clone is left or right distributive by Lemma 4.3, therefore we can apply Theorem 4.15 (after dualizing if necessary) to see that such a groupoid must be a rectangular band, an affine space or (the dual of) a p-cyclic groupoid. This list does not contain any new items compared to Theorem 1.13, thus these two abelianness concepts coincide at the level of abstract minimal clones.

Theorem 4.16. [**Wa2**] *If a minimal clone has a nontrivial weakly abelian representation, then it also has a nontrivial abelian representation. Therefore such a clone must be a unary clone, the clone of an affine space, a rectangular band or a p-cyclic groupoid for some prime p.*

Unary algebras, rectangular bands and affine spaces are abelian. A p-cyclic groupoid must be weakly abelian, as we shall see in the following lemma.

Lemma 4.17. **[Wa2]** *Every p-cyclic groupoid is weakly abelian.*

Proof. Suppose that $\mathbb{A}$ is a p-cyclic groupoid for some prime number p. (Actually, we will not need the fact that p is prime.) Let t be a term of $\mathbb{A}$, of arity $n+m$, and let $\mathbf{a}, \mathbf{b} \in A^n$, $\mathbf{c}, \mathbf{d} \in A^m$ be such that the matrix $\left(\begin{smallmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{smallmatrix}\right)$ is of the form $\left(\begin{smallmatrix} u & u \\ u & v \end{smallmatrix}\right)$. As we have seen in the proof of Lemma 4.8, every term of $\mathbb{A}$ can be reduced to a left-associated product, so we may assume that t is of the form $t = \overleftarrow{x_1 x_2 \ldots x_{n+m}}$. Transposing our matrix if necessary, we can suppose that the leftmost variable is occupied by entries belonging to $\mathbf{a}$ and $\mathbf{b}$, say a_1 and b_1. Using the identity $(xy)z = (xz)y$ we can permute the other variables so that the entries in the first column of the matrix are: $t(\mathbf{a}, \mathbf{c}) = \overleftarrow{a_1 a_2 \cdots a_n c_1 c_2 \cdots c_m}$, and $t(\mathbf{b}, \mathbf{c}) = \overleftarrow{b_1 b_2 \cdots b_n c_1 c_2 \cdots c_m}$. Our groupoid is right cancellative, since multiplication by any element on the right is a permutation of order p. Therefore the equation $t(\mathbf{a}, \mathbf{c}) = t(\mathbf{b}, \mathbf{c})$ implies that $\overleftarrow{a_1 a_2 \cdots a_n} = \overleftarrow{b_1 b_2 \cdots b_n}$. Multiplying both sides on the right by $d_1, d_2, \cdots, d_m$, we conclude that $t(\mathbf{a}, \mathbf{d}) = t(\mathbf{b}, \mathbf{d})$, that is $u = v$, so $\mathbb{A}$ is weakly abelian. $\square$

This lemma yields an interesting homogeneity property for weakly abelian representations.

Theorem 4.18. **[Wa2]** *If a minimal clone has a nontrivial weakly abelian representation, then all representations are weakly abelian.*

As the following example shows, there exist nonabelian p-cyclic groupoids. Therefore the two abelianness concepts differ already at the level of concrete minimal clones.

Example 4.19. For any prime number p let us define the following binary operation on the set $\mathbb{Z}_p \times \{0, 1\}$:

$$(a, b) \circ (c, d) = \begin{cases} (a+1, b) & \text{if } b = 0 \text{ and } d = 1; \\ (a, b) & \text{otherwise.} \end{cases}$$

The algebra $\mathbb{A} = (\mathbb{Z}_p \times \{0, 1\}, \circ)$ is a p-cyclic groupoid, therefore it is weakly abelian and has a minimal clone. It is not abelian, as we can see from the following matrix:

$$\begin{pmatrix} (0,1) \circ (0,0) & (0,1) \circ (0,1) \\ (0,0) \circ (0,0) & (0,0) \circ (0,1) \end{pmatrix} = \begin{pmatrix} (0,1) & (0,1) \\ (0,0) & (1,0) \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

We conclude with a theorem on rectangularity and strong abelianness. A nontrivial affine space or p-cyclic groupoid cannot be rectangular, but unary algebras and rectangular bands are all strongly abelian. Thus these two concepts coincide both at the level of abstract and concrete minimal clones.

Theorem 4.20. **[Wa2]** *If a minimal clone has a nontrivial rectangular representation, then it also has a nontrivial strongly abelian representation; moreover, all representations are strongly abelian. Such a clone must be either unary, or the clone of rectangular bands.*

Chapter 5

Almost associative operations generating a minimal clone

We give two generalizations of Theorem 1.11 in this chapter. In Section 5.1 we discuss two ways to tell how far a given binary operation is from being associative. One of them uses the index of nonassociativity; the other one is based on the associative spectrum. Here we review only some basic facts about them, but in the Appendix we give a more detailed account about the associative spectrum, including many examples. For the index of nonassociativity the reader is referred to [Cl1, Cl2, DK, KT1, Szá].

We call a binary operation almost associative if its associative spectrum or index of nonassociativity is as small as possible without being associative. Thus we have two notions of almost associativity. We study the first one in Section 5.2; it turns out that for idempotent operations the right choice is to require that its associative spectrum $s(n)$ satisfies $1 < s(4) < 5$. The main result of this section is the following characterization of almost associative binary operations generating a minimal clone.

Theorem 5.10 **[Wa3]** *For any groupoid $\mathbb{A}$ the following two conditions are equivalent:*

(i) *$\mathbb{A}$ has a minimal clone and $1 < s_{\mathbb{A}}(4) < 5$;*

(ii) *$\mathbb{A}$ is not a semigroup and $\mathbb{A}$ or its dual belongs to one of the varieties $\mathcal{B} \cap \mathcal{A}$, $\mathcal{C}_p$ or $\mathcal{D} \cap \mathcal{A}$ for some prime p.*

If these conditions are fulfilled, then we have $s_{\mathbb{A}}(n) = 2^{n-2}$ for $n \geq 2$.

In Section 5.3 we consider Szász-Hájek groupoids, i.e. groupoids whose index of nonassociativity equals 1. Szász-Hájek groupoids with minimal clones are described in the next theorem.

Theorem 5.13 **[Wa3]** *For any Szász-Hájek groupoid* $\mathbb{A}$ *the following two conditions are equivalent:*

(i) $\mathbb{A}$ *has a minimal clone;*

(ii) $\mathbb{A}$ *or its dual belongs to the variety* $\mathcal{B}$*.*

Finally, in Theorem 5.14 we will show that there are only 10 Szász-Hájek groupoids up to duality and isomorphism which have a minimal clone, and which are themselves minimal in the sense that all of their proper subgroupoids are semigroups.

5.1 Measuring associativity

One way to measure associativity is to count the *nonassociative triples* in the groupoid; this number (or cardinal) is called the *index of nonassociativity*, and is denoted by ns. Formally, we have $ns\,(\mathbb{A}) = |\{(a,b,c) \in A^3 : (ab)\,c \neq a\,(bc)\}|$. This notion was studied in [Cl1, Cl2, DK, KT1, Szá]. Clearly $\mathbb{A}$ is a semigroup iff $ns\,(\mathbb{A}) = 0$, and it is natural to say that the multiplication of $\mathbb{A}$ is almost associative if $ns\,(\mathbb{A}) = 1$. Such groupoids are called *Szász-Hájek groupoids* (SH-groupoids for short). SH-groupoids were investigated in [Há1, Há2] and [KT3–KT6] in much detail. Following the terminology of these papers, we say that an SH-groupoid is of type (a,b,c) if its only nonassociative triple is $(a,b,c) \in A^3$ and $a \neq b \neq c \neq a$. Types $(a,a,a)\,,(a,b,a)\,,(a,a,b)$ and (a,b,b) are defined analogously. (Note that by saying e.g. that $\mathbb{A}$ is an SH-groupoid of type (a,b,c) we mean not only that the components of the unique nonassociative triple are pairwise distinct, but implicitly we assume that these components are denoted by a, b and c respectively.) Let us recall a result from [KT3] (Proposition 1.2(i)).

Proposition 5.1. **[KT3]** *If* $\mathbb{A}$ *is an SH-groupoid and* (a,b,c) *is the unique nonassociative triple, then* $xy = a$ $(xy = b, xy = c)$ *implies* $x = a$ $(x = b, x = c)$ *or* $y = a$ $(y = b, y = c)$ *for all* $x, y \in A$*.*

Proof. Suppose that $xy = a$, but $x \neq a \neq y$. Since $x \neq a$, we have $(x,y,bc) \neq (a,b,c)$, hence (x,y,bc) is an associative triple: $(xy)\,(bc) = x\,(y\,(bc))$. Now $y \neq a$ implies that $(y,b,c) \neq (a,b,c)$, so $x\,(y\,(bc)) = x\,((yb)\,c)$. Similarly $x\,((yb)\,c) = (x\,(yb))\,c = ((xy)\,b)\,c$, because $x \neq a$. We have obtained that $(xy)\,(bc) = ((xy)\,b)\,c$, thus $(xy,b,c) = (a,b,c)$ is an associative triple, which is a contradiction. The other two assertions can be proved similarly. □

Clearly, a subgroupoid of an SH-groupoid $\mathbb{A}$ with nonassociative triple (a,b,c) is an SH-groupoid or a semigroup, depending on whether it contains a, b and c or not. Specially, $\mathbb{A}$ is generated by $\{a,b,c\}$ iff all proper subgroupoids of $\mathbb{A}$ are semigroups. Such a groupoid is called a *minimal SH-groupoid*. In [KT3-KT6] the project of characterizing minimal SH-groupoids was begun, but completed only for the type (a,a,a). In Theorem 5.13 we prove that SH-groupoids having a minimal clone belong to the varieties $\mathcal{B}$ or $\mathcal{B}^d$, and in Theorem 5.14 we give a complete list of minimal SH-groupoids with a minimal clone up to isomorphism.

Another way of measuring associativity is possible by considering the identities implied by associativity, and somehow counting how many of these are (not) satisfied. To make this more precise, let us say that B is a *bracketing*, if B is a groupoid term, and each variable occurs exactly once in B. If these variables are $x_1, x_2, \ldots, x_n$ and they appear in this order (as we will suppose most of the time), then B is nothing else but a way to put brackets into the product $x_1 \cdot \ldots \cdot x_n$ such that the order of the $n-1$ multiplications is well determined. We express this fact by writing $B = B(x_1, \ldots, x_n)$, and in this case we say that B is of size n. The size of B is denoted by $|B|$.

In every bracketing there is an outermost multiplication, and this splits the bracketing into two parts, the *left factor* and the *right factor* of the bracketing. Let $B = B(x_1, \ldots, x_n)$, and let P, Q be the left and right factors of B. Then $B = PQ$, and $P = P(x_1, \ldots, x_k), Q = Q(x_{k+1}, \ldots, x_n)$, where $k = |P|$. Sometimes we will use the notation $l(B)$ for the left factor of B.

The number of bracketings of the product $x_1 \cdot \ldots \cdot x_n$ is $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$, the $(n-1)$st Catalan number. In a semigroup all of these C_{n-1} many terms induce the same term function, but in an arbitrary groupoid they may induce more than one term function. Intuitively, the more term functions of this kind there are, the less associative the multiplication is. Therefore we define the *associative spectrum* of a groupoid $\mathbb{A}$ to be the sequence $s_{\mathbb{A}}(1), s_{\mathbb{A}}(2), \ldots, s_{\mathbb{A}}(n), \ldots$, where $s_{\mathbb{A}}(n)$ is the number of different term functions on $\mathbb{A}$ arising from bracketings of $x_1 \cdot \ldots \cdot x_n$. Thus the associative spectrum gives (only quantitative) information about identities of the form $B_1(x_1, \ldots, x_n) = B_2(x_1, \ldots, x_n)$ satisfied by the groupoid. The associative spectrum was introduced and investigated in [CsW].

Clearly, $s_{\mathbb{A}}(1) = s_{\mathbb{A}}(2) = 1$ for every groupoid $\mathbb{A}$, and $s_{\mathbb{A}}(3) = 1$ iff $\mathbb{A}$ is a semigroup. In the latter case $s_{\mathbb{A}}(n) = 1$ for all n by the general law of associativity. The smallest possible spectrum for a nonassociative multiplication is $1, 1, 2, 1, 1, \ldots$, so we could say that a binary operation is almost associative if its spectrum is this sequence. However, there is no groupoid having a minimal clone with this spectrum (not even an idempotent one) as we will see later. Therefore we have to be more generous: in Theorem 5.10 we determine groupoids with a minimal clone satisfying $s(4) < 5 = C_3$.

The two ways of measuring associativity introduced here do not seem to be closely related. For example, the groupoid $\mathbb{G}_3$ (see its multiplication table in Table 7) is an SH-groupoid, with the largest possible associative spectrum: $s_{\mathbb{G}_3}(n) = C_{n-1}$ for every n. (For the proof of the latter fact see Proposition A.17; $\mathbb{G}_3$ is isomorphic to the groupoid with number 17 there.)

Let us mention finally that there is a third possibility to measure associativity with the help of the Hamming distance of multiplication tables. This yields the notion of the *semigroup distance* of a groupoid. Groupoids with small semigroup distance and connections between the semigroup distance and the index of nonassociativity were studied in [KT2].

5.2 Minimal clones with small spectrum

In this section we are going to describe nonassociative binary operations generating a minimal clone that have a relatively small associative spectrum. The first three theorems show that the spectrum of such an operation cannot be too small.

Theorem 5.2. [Wa3] *If an idempotent groupoid satisfies the identity*

$$\overleftarrow{x_1 \cdot \ldots \cdot x_n} = \overrightarrow{x_1 \cdot \ldots \cdot x_n} \tag{5.1}$$

for some $n \geq 3$, then it is a semigroup.

Proof. Applying (5.1) with $x_1 = \ldots = x_k = x, x_{k+1} = \ldots = x_n = y$ we obtain

$$xy^{n-k} = \overleftarrow{x \cdot \ldots \cdot x \cdot y \cdot \ldots \cdot y} = \overrightarrow{x \cdot \ldots \cdot x \cdot y \cdot \ldots \cdot y} = {}^{k}xy \tag{5.2}$$

for $1 \leq k \leq n-1$. Let us use (5.1) again, for $x_1 = x$, $x_2 = u = xy^2 = {}^{n-2}xy$, $x_3 = \ldots = x_n = y$:

$$(xu)\, y^{n-2} = \overleftarrow{x \cdot u \cdot y \cdot \ldots \cdot y} = \overrightarrow{x \cdot u \cdot y \cdot \ldots \cdot y} = x\,(uy)\,. \tag{5.3}$$

The left hand side is $(xu)\, y^{n-2} = \left({}^{n-1}xy\right) y^{n-2} = (xy)\, y^{n-2} = xy^{n-1} = xy$ (we used (5.2) twice, with $k = n-1$ and $k = 1$ respectively). We can compute the right hand side of (5.3) in a similar manner: $x\,(uy) = x\,(xy^3) = x\left({}^{n-3}xy\right) = {}^{n-2}xy = xy^2$. Thus we have $xy = xy^2$, i.e. right multiplications are idempotent. Finally, to prove associativity, we write up (5.1) one more time:

$$(xy)\, z^{n-2} = \overleftarrow{x \cdot y \cdot z \cdot \ldots \cdot z} = \overrightarrow{x \cdot y \cdot z \cdot \ldots \cdot z} = x\,(yz)\,.$$

By the idempotence of right multiplication (by z) the left hand side reduces to $(xy)\, z$, and therefore associativity is established. $\square$

Theorem 5.3. **[Wa3]** *An idempotent groupoid satisfying the following two identities for some $n \geq 3$ must be a semigroup.*

$$x_0 \cdot \overleftarrow{x_1 \cdot \ldots \cdot x_n} = x_0 \cdot \overrightarrow{x_1 \cdot \ldots \cdot x_n}$$
$$\overleftarrow{x_1 \cdot \ldots \cdot x_n} \cdot x_0 = \overrightarrow{x_1 \cdot \ldots \cdot x_n} \cdot x_0$$

Proof. Substituting $\overleftarrow{x_1 \cdot \ldots \cdot x_n}$ into x_0 in the first identity we have

$$\overleftarrow{x_1 \cdot \ldots \cdot x_n} = \overleftarrow{x_1 \cdot \ldots \cdot x_n} \cdot \overrightarrow{x_1 \cdot \ldots \cdot x_n}.$$

by idempotence. Similarly, if we substitute $\overrightarrow{x_1 \cdot \ldots \cdot x_n}$ for x_0 in the second identity, then we get

$$\overleftarrow{x_1 \cdot \ldots \cdot x_n} \cdot \overrightarrow{x_1 \cdot \ldots \cdot x_n} = \overrightarrow{x_1 \cdot \ldots \cdot x_n},$$

and thus (5.1), hence also associativity follows by the previous theorem. □

Theorem 5.4. **[Wa3]** *If a groupoid has a minimal clone and satisfies*

$$\overleftarrow{x_1 \cdot \ldots \cdot x_n} = x_1 \cdot \overleftarrow{x_2 \cdot \ldots \cdot x_n} \tag{5.4}$$

for some $n \geq 3$, then it is a semigroup.

Proof. The case $n = 3$ is trivial, so let us suppose that $n \geq 4$. First we draw a consequence of (5.4) and idempotence (putting x and z for x_1 and x_n, and y for the rest of the variables):

$$\left(xy^{n-2}\right) z = x\,(yz)\,. \tag{5.5}$$

As a special case (with $z = y$) we get

$$xy^{n-1} = xy. \tag{5.6}$$

Now we suppose that $\mathbb{A} = (A; \cdot)$ is a groupoid with a minimal clone that satisfies identity (5.4). The binary operation $s\,(x, y) = xy^{n-2}$ belongs to the clone of $\mathbb{A}$, therefore if it is nontrivial, then $[s]$ contains the basic operation $f\,(x, y) = xy$. Suppose that a and b are arbitrary elements of A such that $c = (ab)a^{n-3} \neq a$. We claim that s is a semilattice operation on the two-element set $\{a, c\}$. With the help of (5.6) we see that $s\,(c, a) = \left((ab)a^{n-3}\right) a^{n-2} = (ab)a^{2n-5} = \left((ab)a^{n-1}\right) a^{n-4} = \left((ab)\,a\right) a^{n-4} = (ab)a^{n-3} = c$. To compute $s\,(a, c)$ let us first consider ac:

$$ac = a\left((ab)a^{n-3}\right) = \left((aa)b\right) a^{n-3} = (ab)a^{n-3} = c. \tag{5.7}$$

In the middle two steps we used identity (5.4) and idempotence. Now it is easy to conclude that $s\,(a, c) = ac^{n-2} = c$, proving that s is indeed a semilattice operation on $\{a, c\}$. Since $f \in [s]$, the restriction of f to $\{a, c\}$ is either trivial, or coincides with s. In the latter case we have $f(c, a) = c$, so

$$\left((ab)a^{n-3}\right) a = (ab)a^{n-2} = (ab)a^{n-3}. \tag{5.8}$$

If f is trivial on our two-element set, then it has to be a second projection, because $f(a, c) = ac = c$ as we have already observed in (5.7). Thus we have $f(c, a) = ca = a$, which means that $(ab)a^{n-2} = a$. Multiplying by a from the right we get $(ab)a^{n-1} = a$, therefore $(ab)\,a = a$ by (5.6). If we multiply both sides of this equality $n - 4$ times by a, then we get $(ab)a^{n-3} = a$, i.e. $c = a$, contrary to our assumption. If $(ab)a^{n-3} = a$ for some $a, b \in A$, then (5.8) holds trivially. Thus we have proved that if a groupoid $\mathbb{A}$ has a minimal clone, and satisfies (5.4), then (5.8) holds for all $a, b \in A$. In other words, $\mathbb{A}$ satisfies the following identity.

$$(xy)x^{n-3} = (xy)x^{n-2} \tag{5.9}$$

It suffices to show now that (5.4) and (5.9) together with idempotence imply associativity. Let us multiply both sides of (5.9) by x from the right. We get $(xy)x^{n-2} = (xy)x^{n-1}$ and then (5.6) shows that $(xy)x^{n-2} = (xy)x$. Therefore $((xy)x^{n-2})\,z = ((xy)x)\,z$ also holds. The left hand side of this identity reduces to $(xy)\,(xz)$ according to (5.5), with xy, x and z playing the role of x, y and z, respectively. Thus we have obtained the following identity.

$$((xy)\,x)\,z = (xy)\,(xz) \tag{5.10}$$

Now we go back to (5.9), and this time we multiply it by y from the left. The left hand side becomes $y\,((xy)x^{n-3})$, which turns to $((yx)\,y)\,x^{n-3}$ if we apply (5.4). With the help of (5.10) and idempotence we can simplify this expression: $((yx)\,y)\,x^{n-3} = (((yx)\,y)\,x)\,x^{n-4} = ((yx)\,(yx))\,x^{n-4} = (yx)\,x^{n-4} = yx^{n-3}$. The right hand side of (5.9) becomes $y\,((xy)\,x^{n-2})$. This can be considered as a product of n factors, if we keep the x and the y in the middle together. We can rearrange this product according to (5.4), and we get $(y\,(xy))\,x^{n-2}$. The $y\,(xy)$ at the beginning of this term can be written as $y \cdot \overleftarrow{x \cdot \ldots \cdot x \cdot y}$, and an application of (5.4) yields $\overleftarrow{y \cdot x \cdot \ldots \cdot x \cdot y} = (yx^{n-2})\,y$. Substituting this back into the original expression we get $(y\,(xy))\,x^{n-2} = ((yx^{n-2})\,y)\,x^{n-2}$. If we consider yx^{n-2} as one factor, then this is again a (left-associated) product of n factors, and we can use (5.4) one more time: $((yx^{n-2})\,y)\,x^{n-2} = (yx^{n-2})\,(yx^{n-2})$. Clearly this is just yx^{n-2}, and if we compare the results we have obtained from the two sides of (5.9) we can conclude the following identity.

$$yx^{n-3} = yx^{n-2}$$

Multiplying this by x we get $yx^{n-2} = yx^{n-1} = yx$ by (5.6). Now the left hand side of (5.5) can be simplified as $(xy^{n-2})\,z = (xy)\,z$, and therefore associativity follows. $\square$

Remark 5.5. Idempotence and identity (5.4) for $n \geq 4$ do not imply associativity, as we can see from the following example. For every $k \geq 2$ we define a groupoid $\mathbb{A}_k$ on the set $A_k = \mathbb{Z}_k \dot{\cup} \{e\}$ by

$$xy = \begin{cases} y & \text{if } y \neq e; \\ x+1 & \text{if } y = e \neq x; \\ e & \text{if } y = e = x. \end{cases}$$

This groupoid is idempotent, but not associative, because $(0 \cdot e) \cdot e = 2 \neq 1 = 0 \cdot (e \cdot e)$. Let $B(x_1, \ldots, x_n)$ be a bracketing, b the corresponding term function, and let d_i denote the left depth of x_i in B (see Section A.2 for the definition of left depth). It is not hard to prove by induction on n, that for any $c_1, \ldots, c_n \in A_k$ we have $b(c_1, \ldots, c_n) = c_i + d_i$ if c_i is the last element of the sequence $c_1, \ldots, c_n$ that is different from e (if there is no such element, then clearly $b(c_1, \ldots, c_n) = e$). Thus two bracketings give the same term function on $\mathbb{A}_k$ iff their left depth sequences are congruent modulo k. The left depth sequence of the bracketing on the left hand side of (5.4) is $(n-1, n-2, n-3, \ldots, 1, 0)$ and that of the right hand side is $(1, n-2, n-3, \ldots, 1, 0)$. Hence $\mathbb{A}_k$ satisfies (5.4) iff k divides $n-2$. For example, $\mathbb{A}_{n-2}$ is an idempotent nonassociative groupoid satisfying (5.4).

The associative spectrum of $\mathbb{A}_k$ is the same as that of the operation $x + \varepsilon y$ on $\mathbb{C}$, where ε is a primitive k-th root of unity: both count the number of *zag sequences* modulo k (see Proposition A.30, and the proof of Theorem A.3 for the definition of a zag sequence). If $k = 2$, then we have $\varepsilon = -1$, and the spectrum is 2^{n-2} (cf. Proposition A.4). For $k = 3$ the spectrum is sequence A005773 in the Encyclopedia [Sl]; this sequence is related to Motzkin numbers (A001006). The spectrum for $k = 4$ does not appear in the Encyclopedia, but the superseeker found that it is a transformation of the sequence A036765

Let us now turn to the investigation of four-variable 'associativity conditions'. There are five bracketings of size four:

$$\begin{aligned} B_1 &= x(y(zu)); \\ B_2 &= x((yz)u); \\ B_3 &= (xy)(zu); \\ B_4 &= ((xy)z)u; \\ B_5 &= (x(yz))u. \end{aligned}$$

Many of the possible $\binom{5}{2}$ identities cannot be satisfied by a nonassociative idempotent groupoid. For example, identifying z and u in B_1 and B_3 we see that $B_1 = B_3$ implies associativity if idempotence is assumed. A similar argument works for $B_3 = B_4$ and $B_2 = B_5$. For $B_2 = B_3$ we need two steps: multiplying both sides by a variable from the left yields $x\,(y\,((zu)\,v)) = x\,((yz)\,(uv))$ (after renaming the variables), while replacing u with uv gives $x\,((yz)\,(uv)) = (xy)\,(z\,(uv))$. Now $x\,(y\,((zu)\,v)) = (xy)\,(z\,(uv))$ follows by transitivity, and identifying z, u and v we get $x\,(yz) = (xy)\,z$. We can treat $B_3 = B_5$ similarly (this is actually the dual of $B_2 = B_3$).

Specializing Theorems 5.2 and 5.4 to $n = 4$ we see that $B_1 = B_4$ and $B_2 = B_4$ cannot hold in a nonassociative groupoid with a minimal clone, and neither can $B_1 = B_5$, because it is the dual of $B_2 = B_4$. Only three possibilities remain: our groupoid satisfies $B_1 = B_2$ or $B_4 = B_5$ or both. Theorem 5.3 shows that the third case is impossible, hence we can conclude that if a groupoid $\mathbb{A}$ has a minimal clone, and $1 < s_{\mathbb{A}}(4) < 5$ holds for its spectrum, then $s_{\mathbb{A}}(4) = 4$, and $\mathbb{A}$ satisfies either $B_1 = B_2$ or its dual, but not both. We are going to characterize such groupoids in the next theorem, but first we need three lemmas. Let $\mathcal{A}$ denote the variety defined by $B_1 = B_2$, i.e. $x\,(y\,(zu)) = x\,((yz)\,u)$.

Lemma 5.6. [Wa3] *If $t_1 = t_2$ is an identity that is true in every semigroup, then $\mathcal{A}$ satisfies $xt_1 = xt_2$ (where x is an arbitrary variable).*

Proof. If $t_1 = t_2$ holds in the variety of semigroups, then t_1 and t_2 are two bracketings of the same product. Therefore it suffices to prove that $\mathcal{A}$ satisfies $x \cdot B\,(x_1, \ldots, x_n) = x \cdot \overrightarrow{x_1 \cdot \ldots \cdot x_n}$ for any bracketing $B\,(x_1, \ldots, x_n)$. This is clear for $n = 1, 2$, so let us suppose that $n \geq 3$ and use induction. Repeatedly applying $x\,((yz)\,u) = x\,(y\,(zu))$ we can transform $x \cdot B\,(x_1, \ldots, x_n)$ to the form $x \cdot (x_1 \cdot B'\,(x_2, \ldots, x_n))$. By the induction hypothesis we have that $x_1 \cdot B'\,(x_2, \ldots, x_n) = x_1 \cdot \overrightarrow{x_2 \cdot \ldots \cdot x_n} = \overrightarrow{x_1 \cdot \ldots \cdot x_n}$ holds in $\mathcal{A}$, hence we see that $x \cdot B\,(x_1, \ldots, x_n) = x \cdot \overrightarrow{x_1 \cdot \ldots \cdot x_n}$ is true as well. (Note that we did nothing else but gave a proof for the general law of associativity, but we had to avoid implications of the form $p = q \Rightarrow pr = qr$). $\square$

Lemma 5.7. [Wa3] *Let $\mathcal{V}$ be a subvariety of $\mathcal{A}$, and let $\mathcal{W}$ be the intersection of $\mathcal{V}$ and the variety of semigroups. If an identity $t_1 = t_2$ holds in $\mathcal{W}$, then $xt_1 = xt_2$ holds in $\mathcal{V}$ (where x is an arbitrary variable).*

Proof. Let $\Theta_{\mathcal{V}}, \Theta_{\mathcal{W}}, \Theta_{sgr}$ denote the equational theories of $\mathcal{V}, \mathcal{W}$ and the variety of semigroups, respectively. These are fully invariant congruences of the free groupoid on countably many generators, and $\Theta_{\mathcal{W}}$ equals $\Theta_{\mathcal{V}} \vee \Theta_{sgr}$, i.e. the transitive closure of $\Theta_{\mathcal{V}} \cup \Theta_{sgr}$. Therefore, if $\mathcal{W}$ satisfies an identity $t_1 = t_2$, then there are terms $p_1, \ldots, p_n$ such that $p_1 = t_1$, $p_n = t_2$ and $p_i = p_{i+1}$ holds in $\mathcal{V}$ if i is odd, and $p_i = p_{i+1}$ is a semigroup identity if i is even. Then $xp_i = xp_{i+1}$ is true in $\mathcal{V}$ for every i and any variable x. (For odd i's this is obvious; for even ones it is a consequence of the previous lemma.) Now $xt_1 = xt_2$ follows by transitivity. $\square$

The next lemma is based on the method used in the proof of Lemma 3.8 in [KSz], and is basically just a slight generalization of the situation considered there.

Lemma 5.8. [Wa3] *Suppose that $\mathbb{A}$ is a groupoid with a minimal clone, and there is a subset M of $\mathrm{Clo}^{(2)}(\mathbb{A})$ containing the first projection and at least one nontrivial element, such that for all $f, g, h \in M$ we have*

(i) $f(g, h) = g$;

(ii) $f\left(g, h^d\right) = f(g, e_2) \in M$.

Then $\mathbb{A}$ or its dual belongs to the variety $\mathcal{D}$ or $\mathcal{C}_p$ for some prime number p.

Proof. Let e_1 and e_2 be the first and second binary projection respectively (we can write g^d as $g(e_2, e_1)$ with this notation). Note that $e_2 = e_1^d$, hence (ii) means that $f\left(g, h^d\right)$ does not depend on h (as long as $h \in M$). We have $e_1 \in M$, but $e_2 \in M$ is impossible, because then (ii) would imply (with $f = e_2$) that $h^d = e_2$ for every $h \in M$, contradicting that M has at least two elements. If $f \in M$ is nontrivial and f^d also belongs to M, then we have $f\left(e_1, f^d\right) = e_1$ by (i), and $f\left(e_1, f^d\right) = f(e_1, e_2) = f$ by (ii), hence $f = e_1$, a contradiction. Thus M and $M^d = \left\{f^d : f \in M\right\}$ are disjoint.
The operation $f \bullet g = f(g, e_2)$ is associative in any clone (this is the binary analogue of the corresponding operation introduced in Section 3.1), and $(M; \bullet)$ is a semigroup in virtue of (ii). The first projection is an identity element for $\bullet$, hence $(M; \bullet)$ is a monoid. If N is a submonoid of M, then $N \cup N^d$ is closed under binary compositions and contains e_1 and e_2. In a minimal clone such a set must be either $\{e_1, e_2\}$ or the whole binary part of the clone. This fact together with the disjointness of M and M^d shows that $\mathrm{Clo}^{(2)}(\mathbb{A}) = M \cup M^d$, and the only submonoids of M are $\{e_1\}$ and M itself. Such a monoid is called minimal, and it was shown in Claim 3.11 of [KSz] that every minimal monoid is isomorphic to a two-element semilattice or a cyclic group of prime order.
Suppose first that $(M; \bullet) \cong (\{0, 1\}; \vee)$ with f_0 and f_1 corresponding to 0 and 1 at this isomorphism. Then there are only four binary operations in $\mathrm{Clo}(\mathbb{A})$, namely $f_0 = e_1, f_0^d = e_2, f_1, f_1^d$ and we can suppose (after passing to the dual of $\mathbb{A}$ if necessary) that $f_1(x, y) = xy$, the basic operation in $\mathbb{A}$. By the above isomorphism we have $f_1 = f_{1 \vee 1} = f_1 \bullet f_1 = f_1(f_1, e_2)$, and this means that $xy = (xy)y$ holds in $\mathbb{A}$. Writing out (i) with $f = f_1, g = f_1, h = f_0$ and $f = f_1, g = f_0, h = f_1$ we get $f_1(f_1, f_0) = f_1$ and $f_1(f_0, f_1) = f_0$ implying that $\mathbb{A}$ satisfies the identities $(xy)x = xy$ and $x(xy) = x$. Similarly we obtain $f_1\left(f_0, f_1^d\right) = f_1(f_0, e_2)$ and $f_1\left(f_1, f_1^d\right) = f_1(f_1, e_2)$ as special cases of (ii), and they translate to the identities $x(yx) = xy$ and $(xy)(yx) = (xy)y$. All the identities in Lemma 1.6 are established, therefore $\mathbb{A} \in \mathcal{D}$ follows.
Now let us suppose that $(M; \bullet) \cong (\mathbb{Z}_p; +)$ with $f_i \in M$ corresponding to $i \in \mathbb{Z}_p$ at this isomorphism. Then the binary part of the clone consists of the $2p$ operations f_i, f_i^d $(i = 0, 1, \ldots, p-1)$ with $f_0 = e_1, f_0^d = e_2$. We may assume (after dualizing

if necessary) that $f_i(x, y) = xy$ for some $i \in \{1, \dots, p-1\}$; moreover, we can suppose without loss of generality that $f_1(x, y) = xy$ since the automorphism group of $\mathbb{Z}_p$ acts transitively on $\mathbb{Z}_p \setminus \{0\}$. Then $f_{i+1} = f_1 \bullet f_i = f_1(f_i, e_2)$ by the above isomorphism. Similarly to the previous case, $\mathbb{F}_2(\mathrm{HSP}(\mathbb{A}))$ can be determined: (i) implies $f_i \cdot f_j = f_1(f_i, f_j) = f_i$, and (ii) implies $f_i \cdot f_j^d = f_1\left(f_i, f_j^d\right) = f_1(f_i, e_2) = f_{i+1}$; dualizing these we get $f_i^d \cdot f_j^d = f_i^d$ and $f_i^d \cdot f_j = f_{i+1}^d$. It is easy to check that $\mathbb{F}_2(\mathrm{HSP}(\mathbb{A}))$ is a p-cyclic groupoid with a nontrivial clone (actually it is isomorphic to $\mathbb{F}_2(\mathcal{C}_p)$), hence $\mathrm{HSP}(\mathbb{A}) = \mathcal{C}_p$ by Lemma 1.5. □

Theorem 5.9. **[Wa3]** *Let $\mathcal{V} \subseteq \mathcal{A}$ be a variety with a minimal clone. Then $\mathcal{V}$ or its dual is a subvariety of $\mathcal{B}$, $\mathcal{C}_p$, $\mathcal{D}$ or $\mathcal{RB}$ for some prime p.*

Proof. Let $\mathcal{W}$ be the intersection of $\mathcal{V}$ and the variety of semigroups. Then $\mathcal{W}$ has a minimal or trivial clone, therefore it is a subvariety of the variety of left zero semigroups, right zero semigroups, rectangular bands, left regular bands or right regular bands (cf. Theorem 1.11). We treat these five cases separately.

Case 1. If $\mathcal{W}$ is the variety of left zero semigroups, then Lemma 5.7 shows that $\mathcal{V}$ satisfies $t_1 x = t_1 t$ for arbitrary terms t_1, t if x is the first variable of t. Specializing to $t = t_1$ we have that $\mathcal{V} \models tx = tt = t$, i.e. a $\mathcal{V}$-term does not change if we multiply it by its first variable from the right. Using these observations it is easy to check that $M = \{x, xy, xy^2, xy^3, \dots\}$ satisfies the conditions of Lemma 5.8 for any $\mathbb{A} \in \mathcal{V}$ with a nontrivial clone (especially also for $\mathbb{F}_{\aleph_0}(\mathcal{V})$), and hence $\mathcal{V} \subseteq \mathcal{D}$ or $\mathcal{V} = \mathcal{C}_p$ for some prime p. (Note that $\mathcal{V}$ satisfies $x(yz) = xy$, therefore Lemma 4.8 could be used as well.)

Case 2. If $\mathcal{W}$ is the variety of right zero semigroups, then similarly to the previous case we have the identities $t_1 x = t_1 t$ and $tx = t$ in $\mathcal{V}$, where x is the last variable of t. Now we can apply Lemma 5.8 with $\mathbb{A} = \mathbb{F}_{\aleph_0}(\mathcal{V})$ and $M = \{x, \overleftarrow{xyx}, \overleftarrow{xyxyx}, \overleftarrow{xyxyxyx}, \dots\}$ to show that $\mathcal{V} \subseteq \mathcal{D}$ or $\mathcal{V} = \mathcal{C}_p$ for some prime p, provided $\overleftarrow{xyx}$ is nontrivial in $\mathbb{F}_{\aleph_0}(\mathcal{V})$. If $(xy)x$ is a projection in $\mathbb{F}_{\aleph_0}(\mathcal{V})$, then $\mathcal{V} \models (xy)x = x$ or $\mathcal{V} \models (xy)x = y$. The latter is impossible, since $x((xy)x) = xx = x$ holds in $\mathcal{V}$. Now we can write up the multiplication table of $\mathbb{F}_2(\mathcal{V})$.

$\cdot$	x	y	xy	yx
x	x	xy	xy	x
y	yx	y	y	yx
xy	x	xy	xy	x
yx	yx	y	y	yx

This is a semigroup in $\mathcal{V}$, but it is not a right zero semigroup, contradicting that $\mathcal{W}$ is the variety of right zero semigroups. (Actually this groupoid is isomorphic to the two-generated free rectangular band, hence Lemma 1.5 could be applied as well.)

Case 3. If $\mathcal{W}$ is the variety of rectangular bands, then $\mathcal{V} = \mathcal{W} = \mathcal{RB}$ by Lemma 1.5.

Case 4. Suppose now that $\mathcal{W}$ is a variety of left regular bands. Then $\mathcal{W} \models t_1 = t_2$ if t_1 and t_2 are binary terms such that both x and y appear in both terms, and they have the same first variable. Lemma 5.7 implies that $tt_1 = tt_2$ holds in $\mathcal{V}$ for every term t, if t_1 and t_2 satisfy the above conditions. This allows us to perform the following computations in $\mathcal{V}$ with $g(x,y) = x(xy)$.

$$\begin{aligned}
g(x,g(x,y)) &= x(x(x(xy))) = x(xy) = g(x,y)\\
g(x,g(y,x)) &= x(x(y(yx))) = x(xy) = g(x,y)\\
g(g(x,y),x) &= (x(xy))((x(xy))x) = (x(xy))(x(xy)) = g(x,y)\\
g(g(x,y),y) &= (x(xy))((x(xy))y) = (x(xy))(x(xy)) = g(x,y)\\
g(g(x,y),g(y,x)) &= (x(xy))((x(xy))(y(yx))) = (x(xy))(x(xy)) = g(x,y)
\end{aligned}$$

These identities show that the subclone of $\mathrm{Clo}(\mathcal{V})$ generated by g contains at most four binary operations, namely g, g^d and the two projections. If g is nontrivial, then the minimality of the clone implies that $g(x,y) = xy$ or $g(y,x) = yx$. In the first case the above identities are just the axioms of $\mathcal{B}$, and in the second case they show that $\mathcal{V} \subseteq \mathcal{B}^d$. If g is trivial, then $x(xy) = x$ holds in $\mathcal{V}$ (since $x(xy) = y$ is clearly impossible), and hence also in $\mathcal{W}$. Since $\mathcal{W}$ is a variety of bands, $\mathcal{W} \models x = x(xy) = xy$, and therefore it is the variety of left zero semigroups, and we have Case 1.

Case 5. Finally, let $\mathcal{W}$ be a variety of right regular bands. Now $\mathcal{V} \models tt_1 = tt_2$ whenever the last variable of the binary terms t_1 and t_2 is the same, and the same variables occur in them. Proceeding similarly to the previous case, we show that $[g]^{(2)} = \left\{e_1, e_2, g, g^d\right\}$ for $g(x,y) = x(yx)$. This is established by the following identities.

$$\begin{aligned}
g(x,g(x,y)) &= x((x(yx))x) = x(yx) = g(x,y)\\
g(x,g(y,x)) &= x((y(xy))x) = x(yx) = g(x,y)\\
g(g(x,y),x) &= (x(yx))(x(x(yx))) = (x(yx))(x(yx)) = g(x,y)\\
g(g(x,y),y) &= (x(yx))(y(x(yx))) = (x(yx))(x(yx)) = g(x,y)\\
g(g(x,y),g(y,x)) &= (x(yx))((y(xy))(x(yx))) = (x(yx))(x(yx)) = g(x,y)
\end{aligned}$$

If g is nontrivial, then we have $\mathcal{V} \subseteq \mathcal{B}$ or $\mathcal{V} \subseteq \mathcal{B}^d$ just as in Case 4. If g is trivial, then it has to be a first projection, hence $x(yx) = x$ holds in $\mathcal{V}$. Right regular bands satisfy $x(yx) = yx$, hence $\mathcal{W} \models yx = x$, and we have Case 2. $\square$

Now we are ready to prove the main result of this section: the characterization of groupoids with a minimal clone that are almost semigroups in the 'spectral' sense.

Theorem 5.10. [Wa3] *For any groupoid $\mathbb{A}$ the following two conditions are equivalent:*

(i) *$\mathbb{A}$ has a minimal clone and $1 < s_{\mathbb{A}}(4) < 5$;*

(ii) *$\mathbb{A}$ is not a semigroup and $\mathbb{A}$ or its dual belongs to one of the varieties $\mathcal{B} \cap \mathcal{A}$, $\mathcal{C}_p$ or $\mathcal{D} \cap \mathcal{A}$ for some prime p.*

If these conditions are fulfilled, then we have $s_{\mathbb{A}}(n) = 2^{n-2}$ for all $n \geq 2$.

Proof. First we show that (i) implies (ii). The considerations preceding Lemma 5.6 show that if $\mathbb{A}$ has a minimal clone, and $1 < s_{\mathbb{A}}(4) < 5$, then either $\mathbb{A}$ or its dual satisfies $x(y(zu)) = x((yz)u)$, i.e. $\mathbb{A} \in \mathcal{A}$ or $\mathbb{A} \in \mathcal{A}^d$. Applying Theorem 5.9, we get that $\mathbb{A}$ or $\mathbb{A}^d$ belongs to $\mathcal{B}, \mathcal{C}_p$ or $\mathcal{D}$ (for some prime p). Thus we have to consider varieties of the from $\mathcal{V}_1 \cap \mathcal{V}_2$, where $\mathcal{V}_1 = \mathcal{A}$ or $\mathcal{V}_1 = \mathcal{A}^d$, and $\mathcal{V}_2 \in \left\{\mathcal{B}, \mathcal{C}_p, \mathcal{D}, \mathcal{B}^d, \mathcal{C}_p^d, \mathcal{D}^d : p \text{ is a prime}\right\}$, but up to duality we have only six cases, because we may suppose that $\mathcal{V}_2 = \mathcal{B}, \mathcal{C}_p$ or $\mathcal{D}$.
We show that if $\mathbb{A} \in \mathcal{V}_2$, and a, b are elements of A such that $ax = bx$ holds for all $x \in A$, then $a = b$. Letting $x = a$ and $x = b$ we see that $\{a, b\}$ is a right zero subsemigroup of $\mathbb{A}$. The identity $x(yx) = xy$ holds in $\mathcal{V}_2$ in all of the three cases, hence $a(ba) = ab$. Since a and b form a right zero semigroup we have $a(ba) = a$ and $ab = b$, thus $a = b$ as claimed. We see that $\mathcal{V}_2 \cap \mathcal{A}^d$ is a variety of semigroups, because the defining identity of $\mathcal{A}^d$ is $((xy)z)u = (x(yz))u$, and according to the previous observation this implies that $(xy)z = x(yz)$ holds in $\mathcal{V}_2$. Thus $\mathcal{V}_1 = \mathcal{A}$, and we end up with the varieties of (ii). (Note that $\mathcal{C}_p \models x(y(zu)) = xy = x((yz)u)$, therefore $\mathcal{C}_p \cap \mathcal{A} = \mathcal{C}_p$.)
Now suppose that $\mathbb{A}$ (or its dual) belongs to one of the varieties mentioned in (ii), and $\mathbb{A}$ is not a semigroup. The clone of $\mathcal{B}$, $\mathcal{C}_p$ and $\mathcal{D}$ is minimal, thus the clone of $\mathbb{A}$ is minimal, too (note that $\mathbb{A}$ has a nontrivial clone, because it is not a semigroup). The other assertion of (i) will follow at once, if we prove that $s_{\mathbb{A}}(n) = 2^{n-2}$. We will do this in two steps: first we show that $\mathbb{A} \in \mathcal{A}$ implies $s_{\mathbb{A}}(n) \leq 2^{n-2}$, and then we prove that $s_{\mathbb{A}}(n) \geq 2^{n-2}$ holds if we suppose in addition that $\mathbb{A} \in \mathcal{B}, \mathcal{C}_p$ or $\mathcal{D}$.
Let B and B' be bracketings of the product $x_1 \cdot \ldots \cdot x_n$. Lemma 5.6 implies that $\mathcal{A} \models B = B'$ if $|l(B)| = |l(B')|$ and $\mathcal{A} \models l(B) = l(B')$. Applying Lemma 5.6 again, we see that $|l(B)| = |l(B')|, |l^2(B)| = |l^2(B')|$ and $l^2(B) = l^2(B')$ is sufficient for $B = B'$. Proceeding this way we arrive at left factors of size 1 (i.e. the single variable x_1) finally, and we see that if $|l^i(B)| = |l^i(B')|$ for all i (where it makes sense), then $B = B'$ holds in $\mathcal{A}$. Clearly, the numbers $|l^i(B)|$ (and $|l^i(B')|$) are strictly decreasing in i, therefore it is sufficient if the sets $\{|l^i(B)| : i = 1, 2, \ldots\}$ and $\{|l^i(B')| : i = 1, 2, \ldots\}$ coincide. They are subsets of $\{1, 2, \ldots, n-1\}$, containing 1, hence there are 2^{n-2} many choices for these sets. This shows that $s_{\mathbb{A}}(n) \leq 2^{n-2}$ for any $\mathbb{A} \in \mathcal{A}$.

Now let $\mathbb{A} \in \mathcal{A} \cap \mathcal{V}_2$, where $\mathcal{V}_2 \in \{\mathcal{B}, \mathcal{C}_p, \mathcal{D} : p \text{ is a prime}\}$, and let B and B' be bracketings as before. Suppose that $\mathbb{A} \models B = B'$, but $\{|l^i(B)| : i = 1, 2, \ldots\} \neq \{|l^i(B')| : i = 1, 2, \ldots\}$, and let i be the smallest value where $|l^i(B)|$ and $|l^i(B')|$ are different. The observation made in the second paragraph of this proof (a certain right cancellation property) together with idempotence shows that we can delete the right factors in the identity $B = B'$ if they have the same size. Doing this $i - 1$ times we arrive at bracketings whose left factors have different size, thus we may suppose that $i = 1$ and we can also suppose that $|l^1(B)| < |l^1(B')|$. Let us substitute x for the first $|l^1(B)|$ variables, y for the next $|l^1(B')| - |l^1(B)|$ variables, and z for the rest. Then B becomes $(x \cdots x)(y \cdots yz \cdots z)$ (with some bracketing of the two products), and B' has the form $(x \cdots xy \cdots y)(z \cdots z)$. Thus $\mathbb{A}$ satisfies an identity of the form $(x \cdots x)(y \cdots yz \cdots z) = (x \cdots xy \cdots y)(z \cdots z)$ (with the same number of x, y and z on the two sides).
In $\mathcal{B}$ this identity reduces to $x(yz) = (xy)z$, showing that if $s_{\mathbb{A}}(n) < 2^{n-2}$ for some n, then $\mathbb{A}$ is a semigroup. If $\mathcal{V}_2 = \mathcal{C}_p$ or $\mathcal{D}$, then let us put $y = x$, then we have $\mathbb{A} \models (x \cdots x)(x \cdots xz \cdots z) = (x \cdots xx \cdots x)(z \cdots z)$. The right hand side is clearly xz, and on the left hand side the bracketing of the factor $(x \cdots xz \cdots z)$ is irrelevant according to Lemma 5.6. Thus $\mathbb{A} \models x(xz) = xz$, and since $x(xz) = x$ holds in $\mathcal{C}_p$ and $\mathcal{D}$, we see that $\mathbb{A}$ is a left zero semigroup. We have proved that the associative spectrum of a groupoid in any one of the varieties mentioned in (ii) is either $(1, 1, 1, 1, \ldots)$ or $(1, 2, 4, 8, \ldots)$, and this completes the proof of the theorem. $\square$

Remark 5.11. Each of the varieties $\mathcal{B} \cap \mathcal{A}$, $\mathcal{C}_p$ and $\mathcal{D} \cap \mathcal{A}$ contain groupoids with a nonassociative operation. For $\mathcal{C}_p$ it is clear, because the only p-cyclic groupoids that are semigroups are the left zero semigroups. The two-generated free algebra of $\mathcal{D}$ is not a semigroup, and satisfies $x(y(zu)) = x((yz)u)$, hence belongs to $\mathcal{D} \cap \mathcal{A}$. (See the multiplication table in the proof of Lemma 1.6.) Let us now construct some nonassociative algebras in $\mathcal{B} \cap \mathcal{A}$.

Let $\mathbb{S} = (S; \vee)$ be a semilattice, and let C be the set of finite chains in $\mathbb{S}$. We define a multiplication in C by the following formula (note that if $b_l \leq a_k$, then the right hand side is the same as the first factor on the left hand side).

$$(a_1 < a_2 < \cdots < a_k) \cdot (b_1 < b_2 < \cdots < b_l) = (a_1 < a_2 < \cdots < a_k \leq a_k \vee b_l)$$

For $\mathbf{a} = (a_1 < a_2 < \cdots < a_k)$, $\mathbf{b} = (b_1 < b_2 < \cdots < b_l)$ and $\mathbf{c} = (c_1 < c_2 < \cdots < c_m)$ we have $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} = (a_1 < a_2 < \cdots < a_k \leq a_k \vee b_l \leq a_k \vee b_l \vee c_m)$ and $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}) = (a_1 < a_2 < \cdots < a_k \leq a_k \vee b_l \vee c_m)$. Since the top element of both chains is $a_k \vee b_l \vee c_m$, right multiplication by $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ is the same as right multiplication by $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$, hence $\mathbb{C} = (C; \cdot)$ satisfies $x(y(zu)) = x((yz)u)$. It is not hard to check, that the defining identities of $\mathcal{B}$ also hold in $\mathbb{C}$, hence $\mathbb{C} \in \mathcal{B} \cap \mathcal{A}$. If the height of $\mathbb{S}$ is at least three, i.e. there is a chain of length three, then $\mathbb{C}$ is not a semigroup. Indeed, if $a < b < c$, then $(a \cdot b) \cdot c = (a < b < c) \neq (a < c) = a \cdot (b \cdot c)$.

5.3 Szász-Hájek groupoids with a minimal clone

In this section we are going to determine binary operations generating a minimal clone that are almost associative in the 'index' sense, i.e. SH-groupoids with a minimal clone. We need the following lemma before we state and prove the main result.

Lemma 5.12. **[Wa3]** *If an SH-groupoid has a minimal clone, then it has to be of type* (a, b, c).

Proof. Let $\mathbb{A}$ be an SH-groupoid with a minimal clone. Then $\mathbb{A}$ is idempotent, hence it cannot be of type (a, a, a). If it is of type (a, b, a), then the subgroupoid generated by a and b is a minimal SH-groupoid of type (a, b, a) with a minimal clone. The description of minimal SH-groupoids of type (a, b, a) given in [KT4] is not complete, but it covers the idempotent case (subtypes (α) and (β)). There are four idempotent minimal SH-groupoids of type (a, b, a) up to isomorphism: the following two groupoids and their duals (the second groupoid is a factor of the first one).

$\cdot$	a	b	d	e
a	a	a	e	e
b	d	b	d	d
d	d	d	d	d
e	e	e	e	e

$\cdot$	a	b	d
a	a	a	d
b	d	b	d
d	d	d	d

In both cases the operation $g(x, y) = x(yx)$ is nontrivial, and preserves the equivalence relation corresponding to the partition whose only nontrivial block is $\{b, d\}$, but the basic operation $f(x, y) = xy$ does not preserve this relation. This shows that $f \notin [g]$, hence the clone is not minimal.
Suppose now that $\mathbb{A}$ is of type (a, a, b). From the computations in [KT5] it follows that $d = ba = b$ (combine Lemmas 1.5, 1.6, 2.4 and 2.19), therefore the subgroupoid generated by a and b is a minimal SH-groupoid of type (a, a, b) and of subtype (ε). Up to isomorphism there is only one such groupoid, namely the

following one.

$\cdot$	a	b	c	e
a	a	c	e	e
b	b	b	b	b
c	c	c	c	c
e	e	e	e	e

The clone of this groupoid is not minimal, because $x\,(xy)$ is a nontrivial operation preserving the set $\{a, b, e\}$, while the basic operation xy does not preserve this set. Dually, the type (a, b, b) is not possible either, thus we can conclude that an SH-groupoid with a minimal clone has to be of type (a, b, c). ☐

Theorem 5.13. **[Wa3]** *For any Szász-Hájek groupoid $\mathbb{A}$ the following two conditions are equivalent:*

(i) $\mathbb{A}$ *has a minimal clone;*

(ii) $\mathbb{A}$ *or its dual belongs to the variety $\mathcal{B}$.*

Proof. It is clear that (ii) implies (i), since $\mathcal{B}$ has a minimal clone. For the other direction let us suppose that $\mathbb{A}$ is an SH-groupoid with a minimal clone. As we have seen in the previous lemma, $\mathbb{A}$ is of type (a, b, c). Therefore (x, y, x) is an associative triple for all $x, y \in A$, hence $\mathbb{A} \models (xy)\,x = x\,(yx)$. Thus we may omit parentheses in products of the form xyx. Similarly, we obtain $\mathbb{A} \models (xy)\,y = x\,(yy) = xy$ by idempotence. Proposition 5.1 shows that (xy, x, y) is an associative triple for all $x, y \in A$, because $xy = a, x = b, y = c$ is impossible. Thus $\mathbb{A} \models ((xy)\,x)\,y = (xy)\,(xy) = xy$. By another application of Proposition 5.1 we can see that $(xy, y, x) \neq (a, b, c)$, so $(xy)\,(yx) = ((xy)\,y)\,x = (xy)\,x$ holds in $\mathbb{A}$. The identities derived so far together with their duals are almost sufficient to fill out the multiplication table of the two-generated free algebra in the variety generated by $\mathbb{A}$ (see the table below). The only entries that are not determined yet are $(xyx)\,(yxy)$ and $(yxy)\,(xyx)$. In order to compute these, let us observe that (xyx, yx, y) is always an associative triple, because $yx = b$ and $y = c$ implies $x = b$ by Proposition 5.1, but then $x\,(yx) = bb = b \neq a$. Therefore $\mathbb{A} \models (x\,(yx))\,((yx)\,y) = ((x\,(yx))\,(yx))\,y = (x\,(yx))\,y = ((xy)\,x)\,y = xy$.

$\cdot$	x	y	xy	yx	xyx	yxy
x	x	xy	xy	xyx	xyx	xy
y	yx	y	yxy	yx	yx	yxy
xy	xyx	xy	xy	xyx	xyx	xy
yx	yx	yxy	yxy	yx	yx	yxy
xyx	xyx	xy	xy	xyx	xyx	xy
yxy	yx	yxy	yxy	yx	yx	yxy

We see that the binary part of $\mathrm{Clo}(\mathbb{A})$ contains at most six operations (some of the six elements in the table may coincide). In [LP] we can find the complete description of minimal clones with at most six binary operations, so we could finish the proof by simply examining the list of clones given there. Another way is to observe that for $g(x,y) = xyx$ the binary part of $[g]$ is $\{e_1, e_2, g, g^d\}$. If g is a nontrivial operation, then $[g] = \mathrm{Clo}(\mathbb{A})$, hence $\mathbb{A}$ satisfies $xyx = xy$ or $xyx = yx$, and then the defining identities of $\mathcal{B}$ or $\mathcal{B}^d$ can be read from the above multiplication table. If g is trivial, then $\mathbb{A} \models xyx = x$, because $xyx = y$ would imply $xy = (xyx)\,y = yy = y$. In this case $\mathbb{F}_2(\mathrm{HSP}(\mathbb{A}))$ is a rectangular band (we get the same multiplication table as in Case 2 of the proof of Theorem 5.9), hence $\mathbb{A}$ is a rectangular band by Lemma 1.5, contradicting that $\mathbb{A}$ is an SH-groupoid. $\square$

Finally we describe minimal SH-groupoids in the varieties $\mathcal{B}$ and $\mathcal{B}^d$ up to isomorphism.

Theorem 5.14. **[Wa3]** *Every minimal SH-groupoid having a minimal clone is isomorphic or dually isomorphic to one of the groupoids* $\mathbb{G}_1, \ldots, \mathbb{G}_{10}$ listed *in Table* 7.

Proof. Let $\mathbb{A}$ be a minimal SH-groupoid with a minimal clone. Then $\mathbb{A}$ is of type (a,b,c), and up to duality we may suppose that $\mathbb{A}$ belongs to the variety $\mathcal{B}$. Following the notation of [KT6] we set $d = ab, e = bc, f = a\,(bc) = ae$ and $g = (ab)\,c = dc$. Some of these elements may coincide, but a, b, c are pairwise distinct and $f \neq g$. Since $\mathbb{A}$ is idempotent, we have $d = a$ or $e = c$ by Lemma 1.7 of [KT6]. If $d = a$, then $ba = b$ or $ba = a$ (Lemma 1.9 (iii)); if $e = c$, then $cb = b$ or $cb = c$ (Lemma 1.9 (iv)). Thus we have four cases, and we will deal with them separately.

Case 1. $d = ab = a$ and $ba = b$ We have $g = dc = ac = c$ by Lemma 1.4 (ii) of [KT6], and then $ca = c\,(ca) = (ac)\,(ca) = ac = c$ follows applying the defining identities of $\mathcal{B}$. Some other products may be computed with the help of these identities, for example $be = b\,(bc) = bc = e$ and $eb = (bc)\,b = bc = e$. For others, we can use the fact that (a,b,c) is the only nonassociative triple, e.g.: $cb = (ca)\,b = c\,(ab) = ca = c$, and $bf = b\,(ae) = (ba)\,e = be = e$. We can fill out the multiplication table this way except for the entry fc. Here we have two possibilities. If $e \neq b$, then $(f,e,c) \neq (a,b,c)$, therefore $fc = (fe)\,c = f\,(ec) = fe = f$, and we get the groupoid $\mathbb{G}_1$. If $e = b$, then $fc = ac = c$, and we arrive at the groupoid $\mathbb{G}_3$. (Note that $e = b$ implies $f = ae = ab = a$.) In both cases we have to consider the possibility that some of the elements (denoted by different letters so far) coincide. This amounts to forming factor groupoids, but only with respect to congruences where f and g belong to different congruence classes (otherwise the factor groupoid would be a semigroup). There is no such congruence on $\mathbb{G}_3$, while

$\mathbb{G}_1$ has exactly one nontrivial congruence not collapsing f and $g\,(=c)$; its classes are $\{a\},\{b\},\{c\},\{e,f\}$, and the corresponding factor groupoid is $\mathbb{G}_2$.

Case 2. $d = ab = a$ and $ba = a$ Let us start again with the product ca. We claim that (a, b, ca) is a nonassociative triple. Indeed, $(ab)(ca) = a(ca) = ac = (ab)c = g$, while $a(b(ca)) = a((bc)a) = a(ea) = ae = a(bc) = f$. Since the only nonassociative triple is (a, b, c), we can conclude that $ca = c$. Then $cb = (ca)b = c(ab) = ca = c$, and the rest of the multiplication table can be filled out without any difficulty. (The computation of fc is straightforward here, because $e = b$ is impossible as it would imply $f = bf = ef = e = eg = bg = g$.) We get the groupoid $\mathbb{G}_4$, and the only possible coincidence between the six elements is $e = f$; this yields $\mathbb{G}_5$.

Case 3. $e = bc = c$ and $cb = b$ This case is not possible, because $cb = b$ implies that $b = bb = b(cb)$, but $b(cb) = bc$ by the axioms of $\mathcal{B}$, hence we have $b = bc = c$, which is a contradiction.

Case 4. $e = bc = c$ and $cb = c$ We prove that $cd = c$ by showing that (a, b, cd) is a nonassociative triple. Indeed, $(ab)(cd) = d(cd) = dc = g$, while $a(b(cd)) = f$ can be derived in the following way (we have indicated where we used the axioms of $\mathcal{B}$ and where the Szász-Hájek property).

$$\begin{aligned} a(b(cd)) &\overset{SH}{=} a((bc)d) = a(cd) \overset{SH}{=} (ac)d = (ac)(ab) \\ &\overset{SH}{=} ((ac)a)b \overset{\mathcal{B}}{=} (ac)b \overset{SH}{=} a(cb) = ac = a(bc) = f \end{aligned}$$

Now we can compute that $ca = (cd)a = c(da) = c((ab)a) = c(ab) = cd = c$, and the rest of the multiplication table of $\mathbb{G}_6$ is not hard to fill out (we set $h = ba$ and $i = bf$). The only entries whose calculation is not straightforward are ag, ai and di. Since $f \neq g$, at least one of these two elements is different from c, hence (a, d, f) or (a, d, g) is an associative triple (even if $d = b$). Therefore we have either $ag = a(df) = (ad)f = df = g$, or $ag = a(dg) = (ad)g = dg = g$ (after computing $df = dg = g$ and $ad = d$). Writing ai either as $a(bf)$ or $a(bg)$ and di as $d(bf)$ or $d(bg)$ we get by a similar argument that $ai = g$ and $di = g$. There are four congruences of $\mathbb{G}_6$ that do not collapse f and g, the corresponding factor groupoids are $\mathbb{G}_7, \mathbb{G}_8, \mathbb{G}_9$ and $\mathbb{G}_{10}$.

To finish the proof we need to check that these ten groupoids are really SH-groupoids and that they belong to the variety $\mathcal{B}$. This requires tedious but straightforward computations, therefore we omit the details. ☐

Remark 5.15. Minimal SH-groupoids of type (a, b, c) were investigated in [KT6]. The groupoid $\mathbb{G}_3$ is the same as V_{10} there, but the other nine groupoids found in the previous theorem seem to be new minimal SH-groupoids of type (a, b, c).

Remark 5.16. The class of groupoids found in Theorem 5.10 is disjoint from the class described in Theorem 5.13, i.e. there is no groupoid with a minimal clone that is almost associative in both the 'spectral' and the 'index' sense. Indeed, if $\mathbb{A}$ satisfies the conditions of both theorems, then $\mathbb{A}$ (or its dual) satisfies $x\,(y\,(zu)) = x\,((yz)\,u)$ by the considerations preceding Lemma 5.6, and $\mathbb{A}$ (or its dual) contains a subgroupoid isomorphic to one of the groupoids $\mathbb{G}_1, \ldots, \mathbb{G}_{10}$ by Theorem 5.14. However, this is impossible, because neither of these ten groupoids and neither of their duals satisfy $x\,(y\,(zu)) = x\,((yz)\,u)$ as it can be seen from their multiplication tables (let $x = a, y = a, z = b, u = c$ for $\mathbb{G}_1, \ldots, \mathbb{G}_{10}$ and $x = a, y = c, z = b, u = a$ for their duals).

Appendix

Associative spectra of binary operations

A.1 Introduction

Let n be a positive integer. We call a string consisting of symbols x, (, and) a *bracketing of size* n if it contains n symbols "x", and $n-1$ symbols "(" (left parentheses) as well as ")" (right parentheses) so that they are properly placed to determine a product of n factors x (see, e.g. [BBi, Tam]). More formally,

1. x is the unique bracketing of size 1,
2. the bracketings of size n are exactly the strings of form (PQ) where P and Q are bracketings of size k resp. l with $k+l=n$.

E.g. (xx) is the only bracketing of size 2, and $((x(xx))(xx))$ is a bracketing of size 5. Note that we always use an outermost pair of parentheses whenever $n>1$, in contrary to the everyday usage of parentheses. We shall denote bracketings by capital letters, and $|B|$ stands for the size of B.

Bracketings are, in fact, the elements of the free groupoid with one free generator x (cf. [BBi], p. 133), or, equivalently, they are the unary groupoid terms. The corresponding unary term operations on special groupoids were investigated by several authors (see, e.g. [GN, GS]). In any bracketing of size n we can indicate the position of symbols x by subscripts $1,\ldots,n$, e.g. $(x_1x_2), ((x_1(x_2x_3))(x_4x_5))$. Thus, a bracketing of size n provides also an element of the free groupoid with free generators $x_1,\ldots,x_n$, i.e., an n-ary groupoid term (although, of course, not all n-ary groupoid terms originate from bracketings in such a way). Here we always study bracketings considered as n-ary groupoid terms, even if in some cases we omit the subscripts $1,\ldots,n$. On every groupoid $\mathbb{G}$, these terms give rise to n-ary term operations. We call them *regular n-ary operations of* $\mathbb{G}$ (or, *regular over the operation of* $\mathbb{G}$), and, in concrete cases, *operations induced by given bracketings.*

For notation of the regular operation induced by the bracketing B, P, Q, etc. we use the corresponding lowercase letters b, p, q, etc.

If $\mathbb{G}$ is associative, then by the generalized associative law there is exactly one regular n-ary operation for each n. In the general case, we have a sequence

$$s_{\mathbb{G}}(1), s_{\mathbb{G}}(2), \dots, s_{\mathbb{G}}(n), \dots$$

of positive integers with $s_{\mathbb{G}}(n)$ denoting the number of distinct n-ary regular operations of $\mathbb{G}$. E.g., $s_{\mathbb{G}}(1) = s_{\mathbb{G}}(2) = 1$ for every groupoid $\mathbb{G}$, and $s_{\mathbb{G}}(3) = 2$ if and only if $\mathbb{G}$ is nonassociative, as then the two possible bracketings of size 3, $(x_1(x_2x_3))$ and $((x_1x_2)x_3)$ induce different ternary term operations.

The sequence

$$\{s_{\mathbb{G}}(n)\} = (s_{\mathbb{G}}(1), s_{\mathbb{G}}(2), \dots, s_{\mathbb{G}}(n), \dots)$$

measures, in some sense, the distance of $\mathbb{G}$ from associativity: the smaller its entries are, the closer the operation of $\mathbb{G}$ is to being associative. Hence we call this sequence *the associative spectrum of* $\mathbb{G}$ (or, *of the operation of* $\mathbb{G}$*)*. Instead of $s_{\mathbb{G}}(n)$ we write $s(n)$ if this cannot cause misunderstanding. Usually we also omit $s(1)$ and $s(2)$, bearing no information about $\mathbb{G}$.

In this chapter we study the introduced notion from several points of view. The next section contains some well-known facts, simple observations, and auxiliary results on bracketings and associative spectra; there and later, the routine inductive proofs will often be omitted. Most frequently we use induction on size; we leave out the words "on size" in these cases. The third section contains samples of determining associative spectra of some familiar nonassociative operations. The problem of characterizing all associative spectra of operations on a set with a given power seems to be hard. However, the case of the two-element set is, as it might be expected, easy (Section A.4), and a lot of three-element groupoids are accessible (Section A.5). In the final section we present some facts on the general behavior of associative spectra, and formulate several problems.

Further on, we write simply *spectrum* for associative spectrum.

A.2 Properties of bracketings and spectra

For any bracketing B of size $n(> 1)$, we can *pair* its left and right parentheses in a natural way ([Kl, Tam]). Induction shows that we can always choose a consecutive quadruple (xx) in B; its left and right parentheses will be associated to form a pair. Replacing then (xx) with x we obtain a bracketing B' of size $n-1$, for which the preceding process can be repeated until no unpaired parentheses remain. This way of forming pairs involves that any pair together with the symbols between them is also a bracketing. It is called a *subbracketing of* B; e.g., if $B = (PQ)$, then P and Q are subbracketings of B, as outermost parentheses of any bracketing are paired. We call P and Q the (*left* resp. *right*) *factors of* B. The symbols x are considered as subbracketings of size 1, too. Observe that pairing is unique, and if a parenthesis lies between a pair then its associate also lies between them. Hence the representation of bracketings of size > 1 in form (PQ) is unique, too.

Substituting x for one or several disjoint subbracketings in B we obtain a *quotient bracketing of* B. E.g. $(x(xx))$ and $((xx)(xx))$ are (disjoint) subbracketings of $B = (((x(xx))x)((xx)(xx)))$, and replacing them with x provides the quotient bracketing $((xx)x)$ of B. A bracketing is a *nest* if it is either of size 1 (a *trivial nest*) or one out of its factors is x, and the other one is a nest ([GN, GS]). E.g., all bracketings of size 4 save $(xx)(xx)$ are nests. Given a bracketing B, there are subbracketings of B which are nests; in particular, each x_i is contained in a unique maximal nest. We call these maximal nests simply *the nests of* B. A nontrivial nest has a unique subbracketing of form $(x_i x_{i+1})$; we say that x_i, x_{i+1} are *the eggs of the nest.*

The *Catalan numbers* C_n are defined recursively by

(1) $C_0 = 1$,

(2) $C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0 \quad (n > 0)$,

or, equivalently, by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Comparing (1) and (2) with the formal definition of bracketings in the introduction, and taking into account the uniqueness of the representation of bracketings in form (PQ), we can see that the number of bracketings of size n equals C_{n-1} (see, e.g. [Ja]). Therefore $1 \leq s(n) \leq C_{n-1}$ holds for any spectrum $\{s(n)\}$. If $s_{\mathbb{G}}(n) = C_{n-1}$ for every n, then the groupoid $\mathbb{G}$ and its operation are said to be *Catalan.* E.g., free groupoids are Catalan. The following inequality also follows from the definition of bracketings:

$$s(n) \leq s(1)s(n-1) + s(2)s(n-2) + \cdots + s(n-2)s(2) + s(n-1)s(1) \quad (n \geq 2)\,. \tag{A.1}$$

Hence if $s_{\mathbb{G}}(n_0) < C_{n_0-1}$ then $s_{\mathbb{G}}(n) < C_{n-1}$ for every $n > n_0$.

The spectrum gives account of the number of certain special identities (not) satisfied by a groupoid, therefore isomorphic or antiisomorphic groupoids have the same spectrum. Moreover, for any groupoid $\mathbb{G}$ and $\mathbb{H} \in \mathrm{HSP}\,(\mathbb{G})$ we have $s_{\mathbb{H}}(n) \leq s_{\mathbb{G}}(n)$ for every n. Thus in order to prove that a groupoid $\mathbb{G}$ is Catalan, it is sufficient to find a Catalan groupoid in the variety generated by $\mathbb{G}$.

The next fact goes back to Łukasiewicz (for a proof, see [Co], Ch. 3.2, or [Lo], Exercise 1.38):

Theorem A.1. *Bracketings are uniquely determined by the places of their right (or left) parentheses between the symbols* $x_1, \ldots, x_n$.

Next we introduce sequences of nonnegative integers which arise naturally from bracketings, and also contain full information on them. Consider the free monoid F_2 with unit element e, generated by symbols 0 and 1. A subset M of F_2 is *prefix-free* if no word in M is a prefix (i.e., a left segment) of another word in M. There exist finite maximal prefix-free sets (*FMPF-sets* in short) in F_2, e.g., the set containing the empty word e only, the sets $\{0, 1\}$, $\{00, 010, 011, 10, 11\}$, etc. Assign to each bracketing an ordered sequence of words in F_2 inductively by the rule:

(a) $x \mapsto (e)$,

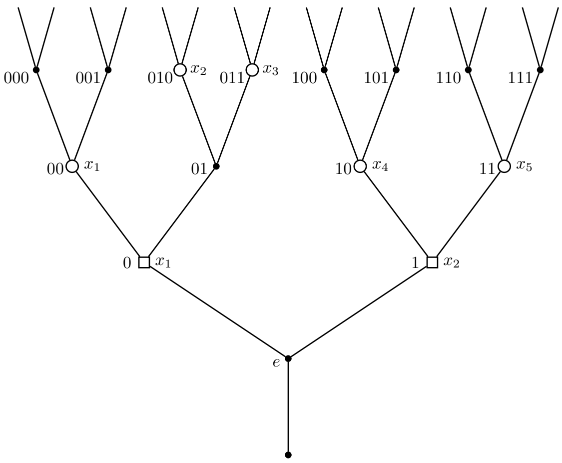
(b) if $P \mapsto (w_1, \ldots, w_k)$ and $Q \mapsto (w_{k+1}, \ldots, w_{k+l})$ then
$$(PQ) \mapsto (0w_1, \ldots, 0w_k, 1w_{k+1}, \ldots, 1w_{k+l}).$$

It is a routine to check that, in this way, a unique, lexicographically listed FMPF-set of n words is assigned to every bracketing of size n. Now we can use the defining properties (1),(2) of Catalan numbers to show that the number of distinct FMPF-sets of n elements equals C_{n-1}. Therefore, (a) and (b) provide a 1-1 correspondence between bracketings and lexicographically ordered FMPF-sets.

Consider a bracketing B of size n viewed with subscripts, i.e., as an n-ary groupoid term. Let $(w_1(B), \ldots, w_n(B))$ be the lexicographically ordered FMPF-set corresponding to B. Call the length of $w_i(B)$ *the depth of* x_i *in* B, and the number of 0's (resp. of 1's) in $w_i(B)$ *the left depth* (resp. *the right depth*) *of* x_i *in* B.

Inspecting (a) and (b) we get the intuitive meaning of depth of x_i: the number of pairs of parentheses (or, equivalently, of the subbracketings of size at least 2) containing x_i. Similarly, e.g. the right depth of x_i in B is the number of those subbracketings in which x_i is contained in the right factor. The sequence consisting of the depths of $x_1, \ldots, x_n$ in B will be called *the depth sequence of* B. *Left* and *right depth sequences of* B are defined analogously. E.g., the depth sequence of $((x(xx))(xx))$ is $(2, 3, 3, 2, 2)$, and its right depth sequence is $(0, 1, 2, 1, 2)$.

FMPF-sets — and thus also bracketings — can be imagined as such minimal sets of vertices in the infinite binary rooted tree that separate the top of the tree from its bottom. See the figure where the sets of vertices corresponding to (x_1x_2) and $(x_1(x_2x_3))(x_4x_5)$ are marked by squares, resp. circles; correspondence between vertices and binary strings is indicated, too. In this representation, the depth of x_i is the number of edges in the path p connecting e with x_i. Similarly, the left (right) depth is the number of left(right) edges in p.



Theorem A.2. **[CsW]** *Bracketings are uniquely determined by their depth sequences.*

Proof. This is clearly true for bracketings of size ≤ 3. Suppose the bracketings (P_1Q_1) and (P_2Q_2) of size $n(> 3)$ have the same depth sequence $(d_1, \dots, d_n)$. From the definition, the equality

$$\sum_{i=1}^{n} \frac{1}{2^{e_i}} = 1 \tag{A.2}$$

follows for every depth sequence $(e_1, \dots, e_n)$. If $|P_1| = j, |P_2| = k$, then, in view of (a) and (b), the depth sequences of P_1 and P_2 are of form $(d_1 - 1, \dots, d_j - 1)$ and $(d_1 - 1, \dots, d_k - 1)$, respectively. Therefore,

$$\sum_{i=1}^{j} \frac{1}{2^{d_i}} = \sum_{i=1}^{k} \frac{1}{2^{d_i}} = 1/2.$$

Hence the sizes of P_1 and P_2 are equal. Now the theorem follows by induction. $\square$

Theorem A.3. **[CsW]** *Bracketings are uniquely determined by their right* (*or left*) *depth sequences.*

Proof. Let $B = (PQ)$ be a bracketing with right depth sequence (in short, *RD-sequence*)

$$(d_1, \dots, d_n). \tag{A.3}$$

Then there is a k between 1 and n such that the RD-sequence of P is $(d_1, \dots, d_k)$, and that of Q is $(d_{k+1} - 1, \dots, d_n - 1)$. Induction shows that always

$$d_1 = 0, \quad d_2 = 1, \tag{A.4}$$

and, for $i = 1, \ldots, n-1$,

$$1 \leq d_{i+1} \leq d_i + 1. \tag{A.5}$$

Call a sequence (A.3) of nonnegative integers a *zag sequence* (cf. [GK], Ch. 1.2, where *zig* is defined) if it has the properties (A.4) and (A.5). We use induction to prove that for any zag sequence (A.3) there exists at most one bracketing with RD-sequence (A.3). This is clearly true for $n \leq 2$. As $(d_{k+1}-1, \ldots, d_n - 1)$ is a zag sequence, we have $d_{k+1} = 1$, and $d_j \geq 2$ for $j = k+2, \ldots, n$. It follows that if the size of the first factor of B is k, then the last 1 in the RD-sequence of B appears on the $(k+1)$st place. Hence if the RD-sequences of $B = (PQ)$ and $B' = (P'Q')$ are the same, then $|P| = |P'|$. Thus the RD-sequences of P and P' coincide, and, by induction, $P = P'$. Similarly we obtain $Q = Q'$, completing the proof. □

An analogous straightforward induction shows that every zag sequence is the RD-sequence of some bracketing. Consequently, the number of zag sequences of length n equals that of the bracketings of size n, i.e., C_{n-1} (cf. [St], Ch. 5, Exercise 19(u)).

A.3 Examples

In this section we determine spectra of several common operations. Given a particular operation, we denote the members of its spectrum by $s(n)$ (without subscript), and we write $s(n) = f(n)$ to indicate that this equality holds for $n \geq 3$.

Proposition A.4. [CsW] *For the subtraction of numbers, $s(n) = 2^{n-2}$.*

Proof. Induction shows that any regular operation $b(x_1, x_2, \ldots, x_n)$ over the subtraction is of form $x_1 - x_2 \pm x_3 \pm \cdots \pm x_n$. It is enough to prove that actually every possible sequence of the $+$ and $-$ signs occurs. This is true for $n \leq 3$; suppose $n > 3$, and apply induction. If $b(x_1, x_2, \ldots, x_n) = x_1 - x_2 - \cdots - x_n$, then b is induced by $((\ldots((x_1x_2)x_3)\ldots)x_n)$. Otherwise there exists a first $+$ sign in f, say $b(x_1, x_2, \ldots, x_n) = x_1 - x_2 - \cdots - x_{k+1} + x_{k+2} \pm \cdots \pm x_{k+l}$ $(k + l = n)$. Then $b(x_1, \ldots, x_n) = (x_1 - x_2 - \cdots - x_k) - (x_{k+1} - x_{k+2} \mp \cdots \mp x_{k+l})$, and this is induced by $B = (PQ)$, where $P = ((\ldots((x_1x_2)x_3)\ldots x_k)$, and Q is the bracketing that induces the subtrahend (such a Q exists by induction). In fact, this reasoning is valid for subtraction in arbitrary Abelian groups except those of exponent 2. □

Proposition A.5. [CsW] *The arithmetic mean as a binary operation on numbers is Catalan.*

Proof. We prove that distinct bracketings induce distinct regular operations over the arithmetic mean. Induction shows that a bracketing B of size n induces $b(x_1, \ldots, x_n) = \sum_{i=1}^{n} 2^{-d_i} x_i$ over the arithmetic mean, where d_i is the depth of x_i

in B. Let $B'\,(\neq B)$ be another bracketing of size n which induces $b'(x_1,\ldots,x_n) = \sum_{i=1}^{n} 2^{-d_i{}'} x_i$. In virtue of Theorem A.2, there exists a j $(1 \leq j \leq n)$ such that $d_j \neq d_j{}'$. Then $b(\delta_1^j,\ldots,\delta_n^j) = 2^{-d_j} \neq 2^{-d_j{}'} = b'(\delta_1^j,\ldots,\delta_n^j)$, i.e., b and b' are distinct operations, as required. This holds for an arbitrary set of numbers closed under arithmetic mean, containing more than one element. ☐

Proposition A.6. **[CsW]** *The geometric mean and the harmonic mean as binary operations on positive real numbers are Catalan.*

Proof. This follows from Proposition A.5 as the groupoids $(\mathbf{R},(x+y)/2)$ and $(\mathbf{R}_+,\sqrt{xy})$ are isomorphic, as well as $(\mathbf{R}_+,(x+y)/2)$ and $(\mathbf{R}_+,2xy/(x+y))$. ☐

Proposition A.7. **[CsW]** *The exponentiation as a binary operation $(a,b)\mapsto a^b$ on numbers is Catalan.*

Proof. Let $p_1,\ldots,p_n$ be distinct prime numbers. Consider bracketings B, $B'(\neq B)$ and the regular operations b,b' they induce over the exponentiation. We show that $b \neq b'$. Making use of the law $(r^s)^t = r^{st}$, and the usual convention of writing r^{s^t} instead of $r^{(s^t)}$, we can write expressions of form $b(p_1,\ldots,p_n)$ without parentheses, e.g., if $B = ((x_1\,(x_2x_3))\,(x_4x_5))$ and p_i are the first primes, we have $b(2,3,5,7,11) = 2^{3^5 7^{11}}$. Here the exponents are at different levels: say, 2 is at the zeroth, 3 and 7 are at the first level, etc. The key observation is that the height of the level of p_i in b always equals the right depth of x_i in B; this can be verified using induction. As $B \neq B'$, by Theorem A.3 there exists a j such that the right depth of x_j in B is different from that of x_j in B'. Then the fundamental theorem of arithmetic implies $b(p_1,\ldots,p_n) \neq b'(p_1,\ldots,p_n)$. ☐

Proposition A.8. **[CsW]** *The cross product of vectors is Catalan.*

Proof. Consider three pairwise perpendicular unit vectors, their additive inverses, and the zero vector. They form a groupoid under cross product, and, if we identify the unit vectors with their negatives, we obtain a four-element factorgroupoid $\mathbb{G}$ with Cayley operation table

$\times$	0	u	v	w
0	0	0	0	0
u	0	0	w	v
v	0	w	0	u
w	0	v	u	0

It is enough to prove that this operation is Catalan, because $\mathbb{G} \in \mathrm{HSP}\,(\mathbb{R}^3;\times)$. Let B, B', b, b' be as in Proposition A.7. We shall find nonzero elements $c_1,\ldots,c_n \in G$

such that $b(c_1, \ldots, c_n) = 0 \neq b'(c_1, \ldots, c_n)$. The case $n = 3$ is obvious. The general case needs some preparations:

Claim A.9. *Let F be a nontrivial nest of size k which induces the regular operation f on $\mathbb{G}$. Given i $(1 \leq i \leq k)$, and $c, d \in G$ with $d \notin \{0, c\}$, we can choose elements $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_k \in G$ so that $f(c_1, \ldots, c_{i-1}, c, c_{i+1}, \ldots, c_k) = d$.*

This is valid also for any bracketing B and its induced regular operation b instead of F and f. Indeed, apply Claim A.9 to the nest of B containing x_i, if this nest is nontrivial, and replace this nest by x; while if x_i is a trivial nest, replace the eggs of another nest by x. Then, in both cases, use induction for the quotient bracketing. We remark that this generalized form of Claim A.9 implies that any regular operation over the cross product is surjective (i.e., it maps G^n onto G; in fact, this is the case for all surjective binary operations, cf. Claim A.13).

Claim A.10. *If x_j, x_{j+1} are no eggs of any nest of a bracketing B, we can choose $d_1, \ldots, d_{j-1}, d, d_{j+2}, \ldots, d_k$ in G such that $f(d_1, \ldots, d_{j-1}, d, d, d_{j+2}, \ldots, d_k) \neq 0$.*

Proof. If $B = (PQ)$ with $|P| = k$ and $j + 1 \leq k$, then for suitable elements $d, d_i \in G$ by induction we have $p(d_1, \ldots, d_{j-1}, d, d, d_{j+2}, \ldots, d_k) = e \neq 0$. Now by Claim A.9 there are $d_{k+1}, \ldots, d_n \in G$ such that $q(d_{k+1}, \ldots, d_n) = f \neq 0, e$. Then $b(d_1, \ldots, d, d, \ldots, d_n) = e \times f \neq 0$. The case $k < j$ can be treated in a similar way. Finally, suppose $k = j$. Let us fix $d \neq 0$, and apply Claim A.9 to P and Q with $i = k$ and $i = k+1$, respectively. Then we have elements $d_1, \ldots, d_{k-1}, d_{k+2}, \ldots, d_n \in G$ such that $p(d_1, \ldots, d_{k-1}, d) = e$ and $q(d, d_{k+2}, \ldots, d_n) = f$, where $G = \{0, d, e, f\}$. Thus $b(d_1, \ldots, d, d, \ldots, d_n) = e \times f = d \neq 0$, completing the proof of Claim A.10. $\diamond$

In order to prove Proposition A.8, first suppose that there is an i $(1 \leq i \leq n)$ such that x_i and x_{i+1} are the eggs of a nest of B as well as of B'. Replacing $(x_i x_{i+1})$ by x in B and B', we obtain quotient bracketings B_1 resp. ${B_1}'$ of size $n-1$ with induced regular operations b_1 and ${b_1}'$. By induction, there exist nonzero elements $e_1, \ldots, e_{n-1} \in G$ such that $b_1(e_1, \ldots, e_i, \ldots, e_{n-1}) = 0 \neq {b_1}'(e_1, \ldots, e_i, \ldots, e_{n-1})$. Let $e', e'' \in G$ be distinct, and different from 0 and e_i. Then $e' \times e'' = e_i$, and $b(e_1, \ldots, e_{i-1}, e', e'', e_{i+1}, \ldots, e_{n-1}) = b_1(e_1, \ldots, e_i, \ldots, e_{n-1}) = 0$, and on the other hand $b'(e_1, \ldots, e_{i-1}, e', e'', e_{i+1}, \ldots, e_{n-1}) = b_1'(e_1, \ldots, e_i, \ldots, e_{n-1}) \neq 0$. Now suppose that no nests of B and B' have a common pair of eggs. Let x_j and x_{j+1} be the eggs of a nest of B. Then $b(d_1, \ldots, d_{j-1}, d, d, d_{j+1}, \ldots, d_n) = 0$ for any choice of $d_1, \ldots, d_{j-1}, d, d_{j+2}, \ldots, d_n \in G$. However, as x_j and x_{j+1} are eggs of no nest in B', from Claim A.10 it follows that there is a choice of $d_1, \ldots, d_{j-1}, d_{j+2}, \ldots, d_n$ such that $b'(d_1, \ldots, d_{j-1}, d, d, d_{j+2}, \ldots, d_n) \neq 0$. $\square$

A.4 Groupoids on two-element sets

In what follows we consider operations on finite sets. For uniform treatment, we study groupoids of form $(\mathbf{n}, \circ)$, where $\mathbf{n}$ stands for the set $\{0, 1, \dots, n-1\}$. Each two-element groupoid is isomorphic or antiisomorphic with $(\mathbf{2}, \circ)$, where $x \circ y$ is one of the following seven Boolean functions:

(1) the constant 1 operation;

(2) x (the first projection);

(3) $x \wedge y$ (i.e., $\min(x, y)$);

(4) $x + y \bmod 2$;

(5) $x + 1 \bmod 2$;

(6) $x|y$ (the Sheffer function: "neither x, nor y");

(7) $x \to y$ (implication).

Here (1) — (4) are associative. We determine the spectra of (5) — (7).

Proposition A.11. **[CsW]** *For the operation $x + 1 \bmod 2$, $s(n) = 2$.*

Proof. Indeed, induction shows that for an arbitrary bracketing B of size n and $c_1, \dots, c_n \in \mathbf{2}$, $b(c_1, \dots, c_n) = c_1 + d \bmod 2$, where d is the left depth of x_1 in B. $\square$

Proposition A.12. **[CsW]** *The Sheffer function is Catalan.*

Proof. Recall, that $0|0 = 1$ and $x|y = 0$ otherwise. We shall need some preliminaries.

Claim A.13. *Regular operations over a surjective operation are surjective* (*i.e., they take on all elements of their base sets*).

Claim A.14. *If the Cayley table of a surjective operation $\circ$ has neither two identical columns nor two identical rows, then each variable of any regular operation over $\circ$ is essential.*

Proof. This is obvious for at most binary regular operations. Let $B = (PQ)$, $|B| = n \geq 3$, $|P| = k$. Take a variable x_i of b. We have to prove that there are elements $c_1, \dots, c_{i-1}, u, v, c_{i+1}, \dots, c_n$ in the base set M of the operation $\circ$ such that $b(c_1, \dots, c_{i-1}, u, c_{i+1}, \dots, c_n) \neq b(c_1, \dots, c_{i-1}, v, c_{i+1}, \dots, c_n)$. Without loss of generality, suppose $i \leq k$. Then by induction there exist $c_1, \dots, c_{i-1}, u, v, c_{i+1}, \dots, c_k \in M$ such that $g = p(c_1, \dots, c_{i-1}, u, c_{i+1}, \dots, c_n) \neq p(c_1, \dots, c_{i-1}, v, c_{i+1}, \dots, c_n) = h$.

The rows of g and h in the Cayley table of $\circ$ are not identical, i.e., there is a $d \in M$ such that $g \circ d \neq h \circ d$. Further, by Claim A.13, there are $c_{k+1}, \ldots, c_n \in M$ with $q(c_{k+1}, \ldots, c_n) = d$. Then $b(c_1, \ldots, c_{i-1}, u, c_{i+1}, \ldots, c_n) = g \circ d \neq h \circ d = b(c_1, \ldots, c_{i-1}, v, c_{i+1}, \ldots, c_n)$, which was needed. $\diamond$

Claim A.15. *If $\circ$ fulfils the conditions of Claim* A.14*, then regular operations of distinct arities over $\circ$ cannot be identically equal.*

Proof. Indeed, otherwise the last variable of the regular operation of greater arity could not be essential. $\diamond$

We see that Claim A.13—Claim A.15 apply to the Sheffer function. Let B_1, B_2 be bracketings of size $n\,(\geq 3), B_1 = (P_1Q_1), B_2 = (P_2Q_2)$, and suppose that their induced operations b_1 and b_2 coincide. We have to prove $B_1 = B_2$. This is true for $n = 3$, as $(0|0)|1 = 0 \neq 1 = 0|(0|1)$. Let $n > 3$, and assume $k = |P_1| \leq |P_2| = l$. First we show that, for arbitrary $c_1, \ldots, c_k, \ldots, c_l \in \mathbf{2}, p_1(c_1, \ldots, c_k) = 0$ if and only if $p_2(c_1, \ldots, c_l) = 0$. Let $p_1(c_1, \ldots, c_k) = 0$. By Claim A.13, there exist $c_{k+1}, \ldots, c_n \in \mathbf{2}$ with $q_1(c_{k+1}, \ldots, c_n) = 0$. Hence it follows

$$\begin{aligned} b_1(c_1, \ldots, c_k, c_{k+1}, \ldots, c_n) &= p_1(c_1, \ldots, c_k) \,|\, q_1(c_{k+1}, \ldots, c_n) = 1 = \\ = b_2(c_1, \ldots, c_l, c_{l+1}, \ldots, c_n) &= p_2(c_1, \ldots, c_l) \,|\, q_2(c_{l+1}, \ldots, c_n), \end{aligned}$$

implying $p_2(c_1, \ldots, c_l) = 0$. This reasoning is valid in the opposite direction, too, showing that p_1 identically equals p_2. Now from Claim A.15 we infer $k = l$ and, by induction, $P_1 = P_2$. It remains to establish $Q_1 = Q_2$. Let, once more, $p_1(c_1, \ldots, c_k) = 0$. If $Q_1 \neq Q_2$, then, again by induction, there are $c_{k+1}, \ldots, c_n \in \mathbf{2}$ such that $q_1(c_{k+1}, \ldots, c_n) \neq q_2(c_{k+1}, \ldots, c_n)$. Then

$$b_1(c_1, \ldots, c_n) = 0 \,|\, q_1(c_{k+1}, \ldots, c_n) \neq 0 \,|\, q_2(c_{k+1}, \ldots, c_n) = b_2(c_1, \ldots, c_n),$$

a contradiction. Thus $Q_1 = Q_2$, as required. $\square$

Proposition A.16. **[CsW]** *Implication is Catalan.*

Proof. Instead of implication we can consider the operation $x{*}y$, defined by $0{*}1 = 1$ and $x * y = 0$ otherwise, as $(\mathbf{2}, \rightarrow)$ and $(\mathbf{2}, *)$ are isomorphic. For $*$, the proof of Proposition A.12 can be literally adapted. $\square$

A.5 Groupoids on three-element sets

There are 3330 essentially distinct three-element groupoids in the sense that each three-element groupoid is isomorphic with exactly one of them (see the Siena Catalog [BBu], in which code numbers from 1 to 3330 are given to each of these representatives), therefore a plain survey of their spectra such as in the two-element case seems to be impossible. In this section we determine the spectra of all groupoids on **3** with minimal clones of term operations, and give examples for further spectra.

There exist 12 essentially distinct groupoids on **3** with minimal clones, and each of them is idempotent (see [Cs1]). The operations of an *idempotent* groupoid on **3** may be encoded by the numbers $0, 1, \ldots, 728$ in the following transparent way: let the code of $\circ$ be

$$(0 \circ 1) \cdot 3^5 + (0 \circ 2) \cdot 3^4 + (1 \circ 0) \cdot 3^3 + (1 \circ 2) \cdot 3^2 + (2 \circ 0) \cdot 3 + (2 \circ 1)$$

(see the examples below). The operations of the groupoids on **3** with minimal clones are (or, more exactly, may be chosen as) $0, 8, 10, 11, 16, 17, 26, 33, 35, 68, 178, 624$ (their codes in the Siena Catalog are $80, 102, 105, 106, 122, 125, 147, 267, 271, 356, 1108, 2346$ respectively). It is easy to check that $0, 8, 10, 11$ and 26 are associative. Here we display the Cayley tables of the remaining seven operations:

$$\begin{array}{cccc}
\begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 2&1&2\end{array} &
\begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 2&2&2\end{array} &
\begin{array}{|ccc}\hline 0&0&0\\ 1&1&0\\ 2&0&2\end{array} &
\begin{array}{|ccc}\hline 0&0&0\\ 1&1&0\\ 2&2&2\end{array} \\
16 & 17 & 33 & 35
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{|ccc}\hline 0&0&0\\ 2&1&1\\ 1&2&2\end{array} &
\begin{array}{|ccc}\hline 0&0&2\\ 0&1&1\\ 2&1&2\end{array} &
\begin{array}{|ccc}\hline 0&2&1\\ 2&1&0\\ 1&0&2\end{array} \\
68 & 178 & 624
\end{array}$$

As we apply three different approaches, we parcel our task into three parts.

Proposition A.17. **[CsW]** *The operations* $16, 17$ *and* 178 *are Catalan.*

Proof. Observe that **3** with each of the operations 16, 17 and 178 is a groupoid in which $\{0, 1\}$ is a subgroupoid with two-sided zero element 0, while $\{1, 2\}$ and $\{2, 0\}$ are subgroupoids with left zero elements 1 and 2, respectively. Here and in what follows, the just considered operations will be denoted by circle. Let $B_i = (P_i Q_i)$ $(i = 1, 2)$ be distinct bracketings of size $n\,(\geq 3)$. For $n = 3$, $1 \circ (2 \circ 0) = 1 \circ 2 = 1 \neq 0 = 1 \circ 0 = (1 \circ 2) \circ 0$, i.e., $b_1 \neq b_2$. To prove the same for $n > 3$, first suppose $|P_1| = k < l = |P_2|$. Then

$$b_1(1, \ldots, 1, 2, \ldots, 2, 0, \ldots, 0) = p_1(1, \ldots, 1) \circ q_1(2, \ldots, 2, 0, \ldots, 0) = 1 \circ 2 = 1,$$
$$b_2(1, \ldots, 1, 2, \ldots, 2, 0, \ldots, 0) = p_2(1, \ldots, 1, 2, \ldots, 2) \circ q_2(0, \ldots, 0) = 1 \circ 0 = 0.$$

Thus, we can assume $|P_1| = |P_2| = k$. If $P_1 \neq P_2$, by induction there exist elements $c_1, \ldots, c_k \in \mathbf{3}$ with $g_1 = p_1(c_1, \ldots, c_k) \neq p_2(c_1, \ldots, c_k) = g_2$. Let d be the element of $\mathbf{3}$ that is different from g_1 and g_2. Then $g_1 \circ d \neq g_2 \circ d$ (see the Cayley tables), and hence $b_1(c_1, \ldots, c_k, d, \ldots, d) = g_1 \circ d$ differs from $b_2(c_1, \ldots, c_k, d, \ldots, d) = g_2 \circ d$. It remains to settle the case $Q_1 \neq Q_2$. Again, we can choose elements $c_{k+1}, \ldots, c_n \in \mathbf{3}$ with $h_1 = q_1(c_{k+1}, \ldots, c_n) \neq q_2(c_{k+1}, \ldots, c_n) = h_2$.

Case 17. Here 0 and 2 are left zero elements, whence $c_{k+1} = 1$, and we can assume $h_1 = 0$, $h_2 = 1$. Now $b_1(1, \ldots, 1, c_{k+1}, \ldots, c_n) = 1 \circ 0 = 0 \neq 1 = 1 \circ 1 = b_2(1, \ldots, 1, c_{k+1}, \ldots, c_n)$.

Cases 16 and 178. For distinct elements $h_1, h_2 \in \mathbf{3}$ there exists $e \in \mathbf{3}$ with $e \circ h_1 \neq e \circ h_2$. Hence it follows $b_1(e, \ldots, e, c_{k+1}, \ldots, c_n) \neq b_2(e, \ldots, e, c_{k+1}, \ldots, c_n)$, concluding the proof. □

Proposition A.18. **[CsW]** *The operation* 33 *is Catalan. For* 35 *and* 68 *we have* $s(n) = 2^{n-2}$.

Proof. Consider a groupoid $(G, \circ)$ with idempotent elements $d, e(\neq d), f$ such that

(α) in the Cayley table of $\circ$, d occurs only in its own row;

(β) in the row of e, $e \circ d$ occurs only once;

(γ) f is a right unit element.

First check that $\mathbf{3}$ with $33, 35$ or 68 satisfies these conditions. Now let $B_1 = (P_1 Q_1)$ and $B_2 = (P_2 Q_2)$ be bracketings of size n such that their induced operations over $\circ$ coincide. We prove $p_1 = p_2$. Suppose $k = |P_1| < |P_2| = l$. Then

$$\begin{aligned} b_2(e, \ldots, e, d, \ldots, d) &= p_2(e, \ldots, e) \circ q_2(d, \ldots, d) = e \circ d, \\ b_1(e, \ldots, e, d, \ldots, d) &= p_1(e, \ldots, e) \circ q_1(e, \ldots, e, d, \ldots, d). \end{aligned}$$

By (α) we have $q_1(e, \ldots, e, d, \ldots, d) \neq d$, therefore from ($\beta$) it follows that $b_1(e, \ldots, e, d, \ldots, d) \neq b_2(e, \ldots, e, d, \ldots, d)$. Thus $|P_1| = |P_2|$, and $p_1(c_1, \ldots, c_k) = b_1(c_1, \ldots, c_k, f, \ldots, f) = b_2(c_1, \ldots, c_k, f, \ldots, f) = p_2(c_1, \ldots, c_k)$ holds for arbitrary $c_1, \ldots, c_k \in G$ by (γ), hence $p_1 = p_2$.
Take into account that 33 is surjective, and its Cayley table has no two identical columns. We show that in the case of 33 if $b_1 = b_2$, then $q_1 = q_2$, which together with $p_1 = p_2$ implies via induction that 33 is Catalan. Indeed, suppose that, although $b_1 = b_2$, there exist $c_{k+1}, \ldots, c_n \in \mathbf{3}$ such that $q_1(c_{k+1}, \ldots, c_n) \neq q_2(c_{k+1}, \ldots, c_n)$. Then the columns of these two elements are also distinct, i.e. $c \circ q_1(c_{k+1}, \ldots, c_n) \neq c \circ q_2(c_{k+1}, \ldots, c_n)$ for some $c \in \mathbf{3}$. In virtue of Claim A.13 we can choose $c_1, \ldots, c_k \in \mathbf{3}$ so that $p_1(c_1, \ldots, c_k) = c$. Now $b_1(c_1, \ldots, c_n) =$

$p_1(c_1,\dots,c_k)\circ q_1(c_{k+1},\dots,c_n)\neq p_1(c_1,\dots,c_k)\circ q_2(c_{k+1},\dots,c_n)=b_2(c_1,\dots,c_n)$, a contradiction.

Concerning 35 and 68, observe that in these cases if $u\circ v\neq u\circ w$ then at least one of v and w is a left zero which satisfies (α). We have seen that $b_1=b_2$ implies $p_1=p_2$; now we prove that the converse implication also holds. Suppose not, i.e., there are $c_1,\dots,c_n\in\mathbf{3}$ such that $b_1(c_1,\dots,c_n)=p_1(c_1,\dots,c_k)\circ q_1(c_{k+1},\dots,c_n)\neq p_1(c_1,\dots,c_k)\circ q_2(c_{k+1},\dots,c_n)=b_2(c_1,\dots,c_n)$. Hence, without loss of generality, the element $d=q_1(c_{k+1},\dots,c_n)$ is a left zero, and d does not occur in other rows. We infer that $c_{k+1}=d$, and, as a consequence, $q_2(c_{k+1},\dots,c_n)=d=q_1(c_{k+1},\dots,c_n)$, whence $b_1(c_1,\dots,c_n)=b_2(c_1,\dots,c_n)$, a contradiction. This shows that, for 35 and 68, $s(n)=s(n-1)+\dots+s(2)+s(1)$, and this means $s(n)=2^{n-2}$, as stated. □

Proposition A.19. [**CsW**] *For the operation* 624, $s(n)=\lfloor 2^n/3\rfloor$.

Proof. 624 is actually $2x+2y \bmod 3$ on $\mathbf{3}$. We shall write it in form $-x-y$; our considerations are valid for this operation on numbers, too. An n-ary regular operation (over $-x-y$) is always of form $t(x_1,\dots,x_n)=\pm x_1\pm\dots\pm x_n$. We call such operations *complete linear*. As $x_1-x_2+x_3$ shows, not every complete linear operation is regular. Denote by $\pi(t)$ the number of $+$ signs in a complete linear operation $t=t(x_1,\dots,x_n)$, and call a complete linear t *subregular*, if $\pi(t)\equiv 2n-1 \pmod 3$. The following assertion can be checked immediately:

Claim A.20. *If* t,t_1,t_2 *are complete linear operations such that the equality* $t(x_1,\dots,x_n)=-t_1(x_1,\dots,x_k)-t_2(x_{k+1},\dots,x_n)$ *holds, then every one of* t,t_1,t_2 *is subregular provided the other two of them are subregular.*

Next we characterize the regular operations over $-x-y$.

Claim A.21. *A complete linear operation* $t(x_1,\dots,x_n)$ *is regular over* $-x-y$ *if and only if it is subregular but not of form*

$$x_1-x_2+x_3-\dots+x_n \tag{A.6}$$

(*i.e., not of odd arity with alternating signs and beginning with a* $+$ *sign*).

Proof. Clearly, this is true for $n\le 3$. Suppose that t is regular. Then $t(x_1,\dots,x_n)=-t_1(x_1,\dots,x_k)-t_2(x_{k+1},\dots,x_n)$ with t_1 and t_2 regular. By induction, t_1 and t_2 are subregular, and Claim A.20 implies that t is subregular. If t is regular and it is of form (A.6), then one of t_1 and t_2 — say, t_1 — must be of even arity with alternating signs. However, a complete linear operation t of arity $2m$ with alternating signs cannot be subregular, as $\pi(t)=m\not\equiv 2\cdot 2m-1 \pmod 3$. Hence t_1 is not subregular, a contradiction. Conversely, assume that t is subregular but not regular. We have to prove that t is of form (A.6). We show that the first sign in t is $+$. If not, then $t(x_1,\ldots,x_n) = -x_1 \pm x_2 \pm \cdots \pm x_n = -x_1 - (\mp x_2 \mp \cdots \mp x_n) = -x_1 - t_2(x_2,\ldots,x_n)$, and from Claim A.20 it follows that t_2 is subregular. If, in addition, t_2 is not of form (A.6), then by induction t_2 is regular, hence t is regular, in contrary to the assumption. However, if t_2 is of form (A.6), then

$$\begin{aligned} t(x_1,\ldots,x_n) &= -x_1 - x_2 + x_3 - \cdots + x_{n-1} - x_n = \\ &= -(x_1 + x_2 - x_3 + \cdots - x_{n-1}) - x_n = \\ &= -t_1(x_1,\ldots,x_{n-1}) - x_n, \end{aligned}$$

and here t_1 is regular, implying again the regularity of t. Thus, t starts with a $+$ sign, and it is enough to prove that the signs alternate in t. If not, consider the first two consecutive identical signs in t. Suppose they are $+$; the other case can be treated analogously. Then

$$\begin{aligned} t(x_1,\ldots,x_n) &= x_1 - x_2 + \cdots - x_{2k-2} + x_{2k-1} + x_{2k} \pm \\ &\quad \pm x_{2k+1} \pm \cdots \pm x_n = \\ &= -\left(-x_1 + x_2 - \cdots + x_{2k-2} - x_{2k-1} - x_{2k}\right) - \\ &\quad - \left(\mp x_{2k+1} \mp \cdots \mp x_n\right) = \\ &= -\, t_1(x_1,\ldots,x_{2k}) - t_2(x_{2k+1},\ldots,x_n). \end{aligned}$$

We can check that t_1 is subregular and not of form (A.6), hence regular; further, t_2 is subregular by Claim A.20. As above, supposing that t_2 is not of form (A.6) leads to a contradiction. Hence $t_2(x_{2k+1},\ldots,x_n) = x_{2k+1} - x_{2k+2} + \cdots - x_{n-1} + x_n$, and

$$\begin{aligned} t(x_1,\ldots,x_n) &= x_1 - x_2 + x_3 - \cdots + x_{2k-1} + x_{2k} - x_{2k+1} + \\ &\quad + x_{2k+2} - \cdots + x_{n-1} - x_n = \\ &= -\left(-x_1 + x_2 - x_3 + \cdots - x_{2k-1} - x_{2k} + x_{2k+1} - \right. \\ &\quad \left. - x_{2k+2} + \cdots - x_{n-1}\right) - x_n = \\ &= -\, {t_1}'(x_1,\ldots,x_{n-1}) - x_n. \end{aligned}$$

Here ${t_1}'$ is subregular and not of form (A.6), so it is regular by induction, whence we obtain that t is regular, and this final contradiction proves that a subregular but not regular complete linear operation is of form (A.6). $\diamond$

From Claim A.21 it follows that the number $s(n)$ of the n-ary regular operations over $-x-y$ equals $\sum_k \binom{n}{3k+i} - (n \bmod 2)$, if $n \equiv 2 - i \pmod 3$ $(i = 0,1,2)$. It is known that each of these numbers is equal to $\lfloor 2^n/3 \rfloor$ (see [GK], Ch. 5, Exercise 75). This completes the description of spectra of three-element groupoids with minimal clones. $\square$

The next seven operations are of some interest from various reasons. The first two pairs have the same spectra but with different coincidences of induced regular operations. Fibonacci numbers appear at the fifth one. Nest structure is exploited in the next example, and the last one is related to the Sheffer operation on **2**. These operations are numbered by their codes in the Siena Catalog [BBu]:

$$\begin{array}{c}\begin{array}{|ccc}\hline 0&0&2\\ 0&0&2\\ 2&2&1\end{array}\\ 1066\end{array}\quad \begin{array}{c}\begin{array}{|ccc}\hline 0&0&0\\ 0&0&0\\ 1&0&0\end{array}\\ 10\end{array}\quad \begin{array}{c}\begin{array}{|ccc}\hline 0&0&1\\ 0&0&1\\ 1&1&0\end{array}\\ 405\end{array}\quad \begin{array}{c}\begin{array}{|ccc}\hline 1&1&1\\ 2&2&2\\ 0&0&0\end{array}\\ 3242\end{array}$$

$$\begin{array}{c}\begin{array}{|ccc}\hline 0&0&0\\ 0&1&0\\ 0&0&1\end{array}\\ 79\end{array}\quad \begin{array}{c}\begin{array}{|ccc}\hline 0&0&0\\ 0&1&0\\ 0&1&2\end{array}\\ 82\end{array}\quad \begin{array}{c}\begin{array}{|ccc}\hline 1&0&0\\ 0&2&0\\ 0&0&0\end{array}\\ 2407\end{array}$$

Proposition A.22. **[CsW]** *For the operations* 1066 *and* 10, $s(n) = n - 1$.

Proof. Denote by $t(c_1, \ldots, c_n)$ the number of occurrences of 2 among $c_1, \ldots, c_n$. Concerning 1066, induction shows that, for arbitrary bracketing $B = (PQ)$ with $|B| = n$, $|P| = k$, and $c_1, \ldots, c_n \in \mathbf{3}$,

$$b(c_1, \ldots, c_n) = 2 \text{ if and only if } t(c_1, \ldots, c_n) \text{ is odd,}$$

and

$$b(c_1, \ldots, c_n) = 1 \text{ if and only if both } t(c_1, \ldots, c_k) \text{ and } t(c_{k+1}, \ldots, c_n) \text{ are odd.}$$

As a consequence, $b(c_1, \ldots, c_n) = 0$ iff both $t(c_1, \ldots, c_k)$ and $t(c_{k+1}, \ldots, c_n)$ are even. Hence it follows that two bracketings of equal size induce the same operation if and only if the sizes of their left factors are equal.
In order to manage 10 (which, for this once, will be written as multiplication), we introduce the *priority of a bracketing* B ($\mathrm{pr}(B)$ in sign) for $|B| > 2$ as follows: If $B = (PQ)$ and $|P| > 1$, then $\mathrm{pr}(B) = 0$; if $B = (x_1(x_2(\ldots(x_k(R))\ldots)))$, and $\mathrm{pr}(R) = 0$ or $|R| = 2$, then $\mathrm{pr}(B) = k$. We call the bracketing R *the core of* B. Clearly, if $n > 2$, for every $k = 0, 1, \ldots, n-2$ there exist bracketings of size n with priority k. Hence it is sufficient to prove that two bracketings of size n induce the same regular operation over 10 if and only if they are of the same priority.
"If": $\mathrm{pr}(B) = 0$ implies that b is the constant 0 operation. If $k = n-2$ or $k = n-3$,

then there is only one bracketing B with $\mathrm{pr}(B) = k$. Suppose B_1 and B_2 are of size n with cores R_1, resp. R_2, and $\mathrm{pr}(B_1) = \mathrm{pr}(B_2) = k < n - 3$. Then

$$\begin{aligned} b_1(c_1, \ldots, c_n) &= (c_1(\ldots(c_k \cdot r_1(c_{k+1}, \ldots, c_n))\ldots)) = (c_1(\ldots(c_k \cdot 0)\ldots)) = \\ &= (c_1(\ldots(c_k \cdot r_2(c_{k+1}, \ldots, c_n))\ldots)) = b_2(c_1, \ldots, c_n) \end{aligned}$$

for arbitrary $c_1, \ldots, c_n \in \mathbf{3}$.
"Only if": Let again B_1 and B_2 be bracketings with cores as above, and let $\mathrm{pr}(B_1) = k < l = \mathrm{pr}(B_2)$. Induction on priority shows that bracketings with positive priority induce nonconstant operations over 10. Hence there exist elements $c_{k+1}, \ldots, c_l, c_{l+1}, \ldots, c_n \in \mathbf{3}$ such that $(c_{k+1}(\ldots(c_l \cdot r_2(c_{l+1}, \ldots, c_n))\ldots)) = 1$. For $i = 0, 1$, check the equality $(2(2(\ldots(2 \cdot i)\ldots))) = (k - i) \bmod 2$, where k is the number of occurrences of 2 in the left side, and choose $c_1 = \cdots = c_k = 2$. It follows

$$\begin{aligned} b_1(c_1, \ldots, c_n) &= (c_1(\ldots(c_k \cdot r_1(c_{k+1}, \ldots, c_n))\ldots)) = (c_1(\ldots(c_k \cdot 0)\ldots)) = \\ &= k \bmod 2 \neq (k-1) \bmod 2 = (c_1(\ldots(c_k \cdot 1)\ldots)) = \\ &= (c_1(\ldots(c_k(c_{k+1}(\ldots(c_l \cdot r_2(c_{l+1}, \ldots, c_n))\ldots)))\ldots)) = \\ &= b_2(c_1, \ldots, c_n). \end{aligned}$$

□

Proposition A.23. [CsW] *For the operations* 405 *and* 3242, $s(n) = 3$ *if* $n > 3$.

Proof. Let B_1, B_2 be bracketings of size n, $B_i = (P_i Q_i)$. We show that the induced regular operations b_1, b_2 over 405 coincide if and only if one of the following conditions is satisfied:

(1) $|P_1| = |P_2| = 1$;

(2) $1 < |P_1|, |P_2| < n - 1$;

(3) $|P_1| = |P_2| = n - 1$.

Indeed, in the case (1) the first variable, and in the case (3) the last variable determines the value of b_i. In the case (2) b_i is the constant zero operation. Finally, if $B_1 = (x_1 Q_1)$, $B_2 = (P_2 x_n)$, then $b_1(0, \ldots, 2) = 0 \neq 1 = b_2(0, \ldots, 2)$. 3242 is $x + 1 \bmod 3$. Similarly to Proposition A.11, for any bracketing B and its induced operation b over 3242 we have $b_1(c_1, \ldots, c_n) = c_1 + d \bmod 3$, where d is the left depth of x_1 in B. □

Proposition A.24. [CsW] *For the operation* 79, $s(n) = F_{n+1} - 1$, *where* F_k *is the* k*th Fibonacci number.*

Proof. First we show that, for bracketings B_1, B_2 of equal size, b_1 coincides with b_2 if and only if the eggs of nests of B_1 are the same as the eggs of nests of B_2. Suppose that x_i, x_{i+1} are the eggs of a nest of B_1 but of no nest of B_2. Put $c_j = 2$, if $j = i$ or $j = i + 1$, and $c_j = 1$ otherwise. Then $b_1(c_1, \dots, c_n) = 1 \neq 0 = b_2(c_1, \dots, c_n)$. On the other hand, if the eggs of nests of B_1 and B_2 are the same, induction on the number of nests proves $b_1 = b_2$. Note that this number is 1 exactly when B_1 and B_2 are nests, and for nests we can apply the usual induction on size. Choose several non-overlapping pairs $(i, i + 1)$ in the sequence $1, \dots, n$. The number of such choices (including the empty choice) is F_{n+1}. Induction shows that for every such nonempty choice C there exists a bracketing B such that x_i, x_{i+1} are the eggs of a nest of B if and only if $(i, i + 1)$ occurs in the choice C. This proves our proposition. $\square$

Proposition A.25. [CsW] *The operation* 82 *is Catalan.*

Proof. Induction shows that the first (i.e., leftmost) right parenthesis in B together with its left pair encloses just the eggs of the leftmost nontrivial (maximal) nest of B. Let $|B_1| = |B_2| = n$, $b_1 = b_2$, and let the eggs in question of B_1 and B_2 consist of x_k, x_{k+1} and x_l, x_{l+1} $(k < l)$, respectively. For $c_1 = \dots = c_k = c_{k+2} = \dots = c_n = 1$, $c_{k+1} = 2$ we get $b_1(c_1, \dots, c_n) = 0 \neq b_2(c_1, \dots, c_n)$. Thus, the first right parentheses in B_1 and B_2 cannot be in different positions. Collapsing x_k and x_{k+1} we obtain quotient bracketings B_1' and B_2' of size $n - 1$. Remark that, for arbitrary $c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n \in \mathbf{3}$, $b_i'(c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n) = b_i(c_1, \dots, c_{k-1}, 2, c_{k+1}, \dots, c_n)$ holds, as 2 is a left unit for 82. In such a way, b_i determines b_i', and the latter determines the place of the first right parenthesis in B_i', which is the second right parenthesis in B_i; etc. We see that the induced operation determines the positions of all right parentheses in its parent bracketing. Now Proposition A.25 follows from Theorem A.1. $\square$

Proposition A.26. [CsW] *The operation* 2407 *is Catalan.*

Proof. The proof consists of a suitable adaptation of Proposition A.12. The observations Claim A.13, Claim A.14, and Claim A.15 apply to 2407. Now, from $B_1 = (P_1Q_1)$, $B_2 = (P_2Q_2)$, and $b_1 = b_2$ we can deduce not only the equivalence of $p_1(c_1, \dots, c_k) = 0$ and $p_2(c_1, \dots, c_l) = 0$ but also that of $p_1(c_1, \dots, c_k) = 1$ and $p_2(c_1, \dots, c_l) = 1$. Thus, again we have $p_1 = p_2$, and, by induction, $P_1 = P_2$. In order to refute $Q_1 \neq Q_2$, assume that there exist $c_{k+1}, \dots, c_n \in \mathbf{3}$ with $q_1(c_{k+1}, \dots, c_n) = i \neq j = q_2(c_{k+1}, \dots, c_n)$; here we can suppose $i \neq 2$. There are $c_1, \dots, c_k \in \mathbf{3}$ with $p_1(c_1, \dots, c_k) = i$. Then $b_1(c_1, \dots, c_n) = i \circ i = i + 1 \bmod 3 \neq i \circ j = b_2(c_1, \dots, c_n)$. $\square$

The Sheffer function on **2** and 2407 on **3** are the smallest instances of groupoids $(\mathbf{n}, \circ)$ with operations

$$i \circ j = \begin{cases} i+1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases} \tag{A.7}$$

All these groupoids are *primal*; i.e., all possible operations on **n** are term operations of such a groupoid. The proof of Proposition A.26 can be generalized for them without trouble. Hence we could (in fact, we did) conjecture for a minute that primality implies a Catalan spectrum; however, operation 3233 testifies that this is not the case. Its Cayley table comes from that of 3242 by writing $1 \circ 2 = 0$ instead of $1 \circ 2 = 2$. For 3233 we have $s_6 = 41 < C_5 (= 42)$. Actually,

$$x_1 \circ ((x_2 \circ (x_3 \circ (x_4 \circ x_5))) \circ x_6) = x_1 \circ ((x_2 \circ ((x_3 \circ x_4) \circ x_5)) \circ x_6)$$

identically holds for 3233 on **3** (but no other regular operations over 3233 induced by distinct bracketings of size ≤ 6 are equal). On the other hand, the primality of **3** with 3233 as well as of **n** with operation (A.7) follows, e.g., from Rousseau's criterion: a finite algebra with a single operation is primal if and only if it has neither proper subalgebras, nor congruences, nor automorphisms [Rou].

We have checked all the 3330 entries of the Siena Catalog by computer for the five initial elements of their spectra, i.e. $(s(3), s(4), s(5), s(6), s(7))$. It is known that there are 24 nonisomorphic three-element semigroups. The table below shows the number of essentially distinct three-element nonassociative groupoids with a given initial segment of spectrum:

2	2	2	2	2	16
2	3	3	3	3	4
2	3	4	5	6	15
2	4	4	4	4	2
2	4	5	6	7	6
2	4	6	8	10	4
2	4	7	12	20	4
2	4	7	12	21	12
2	4	8	15	27	12
2	4	8	16	32	62
2	5	8	12	16	2
2	5	10	18	31	4
2	5	10	20	40	4
2	5	10	21	42	5
2	5	11	23	47	2
2	5	11	24	53	4
2	5	12	28	65	12
2	5	13	34	87	12
2	5	13	34	89	2
2	5	13	34	90	4
2	5	13	34	91	24
2	5	13	35	96	2
2	5	13	35	97	32
2	5	14	41	123	6
2	5	14	41	124	16
2	5	14	42	132	3038

Several sequences beginning with some quintuples above, e.g. $(2, 5, 10, 21, 42)$ (cf. Proposition A.19) and $(2, 5, 14, 41, 123)$, are recently missing in the Encyclopedia [Sl].

A.6 General remarks and problems

All the spectra considered up to now are monotonic. Groups with the commutator operation provide examples of non-monotonic spectra: if a group G is nilpotent then there exists an n such that all n-ary regular term operations over the commutator of G are equal (to the constant unit operation), hence $s(n) = 1$, and if G is not nilpotent of class 2 then the commutator is not associative (see, e.g. [Ku]). The spectrum always stabilizes in these examples: $s(n) = 1$ implies $s(m) = 1$ for every $m > n$. In fact, this is a common property of all spectra, which generalizes the generalized associative law:

Theorem A.27. **[CsW]** *For an arbitrary spectrum s, $s(n) = 1$ for some $n\,(\geq 3)$ implies $s(m) = 1$ for every $m > n$.*

Proof. Call two bracketings of size m *adjacent* if there exists a j such that x_j, x_{j+1} are eggs of nests for each of these bracketings. It is easy to see that the transitive closure of the adjacency relation is the trivial equivalence if $m \geq 5$. Let $n\,(\geq 3)$ be a number such that $s(n) = 1$ for an operation $\circ$ on a set M. Consider bracketings B, B^* of size $n + 1$. We have to prove $b = b^*$. For $n = 3$ this is the generalized associative law. Assume $n > 3$. Then $n + 1 \geq 5$, hence there exist bracketings $B_0 = B, B_1, \ldots, B_k = B^*$ such that, for $i = 0, 1, \ldots, k - 1$, B_i is adjacent to B_{i+1}. Let x_j, x_{j+1} be common eggs of a nest of B_i and a nest of B_{i+1}. Replacing $(x_j x_{j+1})$ by x_j in both of them, we obtain quotient bracketings B'_i, B'_{i+1} of size n. As $s(n) = 1$, we have $b'_i = b'_{i+1}$, and thus

$$\begin{aligned} b_i(c_1, \ldots, c_{n+1}) &= b'_i(c_1, \ldots, c_{j-1}, c_j \circ c_{j+1}, c_{j+2}, \ldots, c_{n+1}) = \\ &= b'_{i+1}(c_1, \ldots, c_{j-1}, c_j \circ c_{j+1}, c_{j+2}, \ldots, c_{n+1}) = \\ &= b_{i+1}(c_1, \ldots, c_{n+1}) \end{aligned}$$

for arbitrary $c_1, \ldots, c_{n+1} \in M$. □

Groups provide also examples showing that the difference $s(n) - s(n - 1)$ of consecutive entries of a spectrum can be arbitrarily large:

Proposition A.28. **[CsW]** *The spectrum of the commutator operation on the dihedral group of degree 2^t $(t \geq 3)$ is*

$$s(n) = \begin{cases} 2, & \textit{if } n = 3 \\ n, & \textit{if } 3 < n \leq t \\ 1, & \textit{if } n > t. \end{cases}$$

Proof. D_m, the dihedral group of degree m is generated by a rotation α of order m and a reflection ρ. We write i for α^i and j' for $\alpha^j\rho$. Here is the concise Cayley table of the commutator on D_m:

	j	j'
i	0	$-2i \bmod m$
i'	$2j \bmod m$	$2\,(i-j) \bmod m$

The following observations are immediate: If a bracketing B over the commutator on D_n has at least two nests, then it induces the constant zero operation. Further, if B is a nest with eggs x_k, x_{k+1}, then $b(c_1,\dots,c_n) \neq 0$ only if all c_i $(\in D_m)$ but at most one of c_k, c_{k+1} are of form i' (i.e., $\alpha^i\rho$). From the Cayley table we learn that for such a nest B and such elements $c_1,\dots,c_n$

$$b(c_1,\dots,c_n) = [c_k, c_{k+1}]\, 2^{k-1}(-2)^{n-k-1} \bmod m \tag{A.8}$$

holds. From (A.8) we infer that the position of eggs of B determines the induced operation b. As all commutators are of form $2u \bmod m$, (A.8) shows also that always $b(c_1,\dots,c_n) = 2^{n-1}\cdot v \bmod m$ with suitable integers v. This means that b is the zero operation if $m = 2^t$ and $n > t$.
It remains to show that nests of equal size $n\,(\leq t)$ but with distinct eggs induce distinct operations. Indeed, besides B consider another nest B' with eggs x_l, x_{l+1} $(l > k)$. Let $c_k = 1$, $c_{k+1} = 2'$, and choose elements c_i $(i \neq k, k+1)$ of form i' arbitrarily. Then $[1,2'] = -2 \bmod 2^t$, and, by (A.8), $b(c_1,\dots,c_n) = (-1)^{n-k}2^{n-1} \bmod 2^t \neq 0$. On the other hand, $l > k$ implies $b'(c_1,\dots,c_n) = 0$ because $c_k = 1$, and x_k is out of the egg of B'. ☐

The same reasoning shows that the commutator on D_1, D_2 and D_4 is associative, and if m is not a power of 2 (e.g., in the case of $D_3 = S_3$) the spectrum of the commutator on D_m is $s(n) = n$ for $n > 3$.

The next example leads to groupoids whose spectra begin with arbitrarily many Catalan numbers and still reach 1.

Proposition A.29. **[CsW]** *The following operation on the nonnegative integers is Catalan:*

$$a \circ b = \begin{cases} \min(a,b) - 1, & \text{if } a, b > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For the proof, denote by $d_B(x_i)$ the depth of x_i in the bracketing B. Consider an arbitrary bracketing $B = (PQ)$ with $|B| = n$, $|P| = k$. First we show that

$$b(d_B(x_1)+1,\dots,d_B(x_n)+1) = 1.$$

Note that, for any B, $b(c_1, \ldots, c_n) > 0$ implies $b(c_1+1, \ldots, c_n+1) = b(c_1, \ldots, c_n)+1$. By induction we have $p(d_B(x_1), \ldots, d_B(x_k)) = p(d_P(x_1) + 1, \ldots, d_P(x_k) + 1) = 1$, and similarly $q(d_B(x_{k+1}), \ldots, d_B(x_n)) = 1$, whence it follows

$$\begin{aligned} b(d_B(x_1) + 1, \ldots, d_B(x_n) + 1) &= p(d_B(x_1) + 1, \ldots, d_B(x_k) + 1)\circ \\ &\quad \circ q(d_B(x_{k+1}) + 1, \ldots, d_B(x_n) + 1) = \\ &= (1 + 1) \circ (1 + 1) = 1, \end{aligned}$$

as needed. Next we show that for any other B' of size n we have $b'(d_B(x_1) + 1, \ldots, d_B(x_n) + 1) = 0$. Again, induction shows that for arbitrary B, nonnegative integers $c_1, \ldots, c_n$, and i $(1 \le i \le n)$

$$b(c_1, \ldots, c_n) \le \max(c_i - d_B(x_i), 0) \tag{A.9}$$

holds; we omit the details. As $B' \ne B$, Theorem A.2 implies that there exists an i such that $d_{B'}(x_i) \ne d_B(x_i)$, and in view of (A.2) we can suppose even $d_{B'}(x_i) > d_B(x_i)$. Then applying (A.9) to B' we obtain

$$b'(d_B(x_1) + 1, \ldots, d_B(x_n) + 1) \le \max(d_B(x_i) + 1 - d_{B'}(x_i), 0) = 0,$$

concluding the proof. $\square$

For any bracketing B with $|B| = k < n$, and for every $i\,(= 1, \ldots, k)$, we have $d_B(x_i) < k$, hence $d_B(x_i) + 1 \in \mathbf{n}$. Therefore the above reasoning shows that in $(\mathbf{n}, \circ)$, which is a subgroupoid of $(\mathcal{N}_0, \circ)$, distinct bracketings of size $k\,(< n)$ induce different regular operations. On the other hand, every bracketing B whose size exceeds 2^{n-2} has a symbol x_j with $d_B(x_j) \ge n - 1$. Applying (A.9) to the regular operation b of $(\mathbf{n}, \circ)$ we obtain

$$b(c_1, \ldots, c_n) \le \max(c_j - d_B(x_j), 0) = 0,$$

as $c_j \le n - 1$. Hence any bracketing of size $2^{n-2} + 1$ induces the constant zero operation of $(\mathbf{n}, \circ)$. Thus, for the spectrum of $(\mathbf{n}, \circ)$, $s(k) = C_{k-1}$ if $k < n$, and $s(k) = 1$ if $k > 2^{n-2}$.

The study of spectra of linear operations $px + qy$ (and $px + qy + r$) on numbers (or, more generally, on modules over rings) also offers remarkable facts. As a specimen, we prove the following generalization of Proposition A.5.

Proposition A.30. [**CsW**] *The linear operations $px + py$ and $x + py$ on the complex numbers are not Catalan if and only if p is a root of unity.*

Proof. Concerning $px + py$, induction shows that for any bracketing B of size n, the induced operation over $px + py$ is

$$b(x_1, \dots, x_n) = \sum_{i=1}^{n} p^{d_i} x_i, \tag{A.10}$$

where d_i is the depth of x_i in B. From Theorem A.2 it follows that if p is not a root of unity then $px + py$ is Catalan. Suppose $p^k = 1$. Define the bracketings B_i by $B_1 = (xx)$, and $B_{n+1} = (B_n B_n)$ for $n > 0$. The depth sequences of $B' = (xB_k)$ and $B'' = (B_k x)$ are $(1, k+1, \dots, k+1)$ and $(k+1, \dots, k+1, 1)$, respectively. Now (A.10) implies $b' = b''$. Hence, for $m = 2^k + 1$, $s(m) < C_{m-1}$. Analogous considerations apply to $x + py$: (A.10) remains valid for this case with *right* depths instead of depths. If p is not a root of unity, Theorem A.3 guarantees that $x + py$ is Catalan. Suppose again $p^k = 1$, and redefine B_i by $B_1 = (xx)$, and $B_{n+1} = (xB_n)$ for $n > 0$. The RD-sequences of $B' = (B_k x)$ and $B'' = B_{k+1}$ are $(0, 1, 2, \dots, k, 1)$ and $(0, 1, 2, \dots, k, k+1)$, $\square$

In conclusion, we formulate a few problems:

1. For every positive integer n there exists a minimal $f(n)$ with the property that, if for two spectra s_1, s_2 of n-element groupoids $s_1(i) = s_2(i)$ holds whenever $i \leq f(n)$, then these spectra coincide. Propositions A.11—A.16 imply $f(2) = 4$, and the table at the end of Section A.5 shows that $f(3) \geq 7$. What is the actual value of $f(3)$ (and that of $f(4)$, etc.)?

2. We gave a rough estimation for the subsequent entries of a spectrum with a given initial segment in (A.1) which e.g., for $s(3) = 2$ and $s(4) = 4$ provides $s(5) \leq 12$. However, a case-by-case analysis shows that $s(3) = 2$ and $s(4) = 4$ actually imply $s(5) \leq 8$. Do they imply $s(n) \leq 2^{n-2}$ for all $n\,(> 1)$? If so, call $s(n) = 2^{n-2}$ a *maximal extension* of the initial segment $(2, 4)$. Prove or disprove that the maximal extension of $(2, 3)$ is $s(n) = n - 1$, and that of $(2, 2)$ is $s(n) = 2$.

3. All nonconstant spectra we exhibited above are ultimately constant or monotonic. In the latter case their growth rates are either linear or exponential. Is there any other possibility? More concretely: find, e.g., a spectrum with quadratic growth rate.

4. The statistics of the three-element groupoids and the abundance of appropriate examples leave such an impression that a huge majority of binary operations is Catalan. Is it true that, in some sense, almost all operations are Catalan (or almost Catalan)?

Summary

1. Introduction

A *(concrete) clone* is a collection $\mathcal{C}$ of finitary operations on a set that is closed under composition of functions and contains all projections. An *(abstract) clone* is a heterogeneous algebra that captures the compositional structure of concrete clones. A *representation* of an abstract clone $\mathcal{C}$ is (the image of) a clone homomorphism from $\mathcal{C}$ to the concrete clone of operations on some set. The most important examples of clones are clones of term functions of algebras.

A clone is *minimal* if its only proper subclone is the trivial clone, i.e. the clone of projections. Clearly, a nontrivial clone is minimal iff it is generated by any of its nontrivial elements. It is convenient to choose a function of the least possible arity as a generator of a minimal clone. These generators are called *minimal functions*. A minimal function must be of one of five types according to the following theorem of I. G. Rosenberg.

Theorem 1. **[Ros]** *Let f be a nontrivial operation of minimum arity in a minimal clone. Then f satisfies one of the following conditions:*

(I) *f is unary, and $f^2(x) = f(x)$ or $f^p(x) = x$ for some prime p;*

(II) *f is a binary idempotent operation, i.e. $f(x, x) = x$;*

(III) *f is a ternary majority operation, i.e. $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$;*

(IV) *$f(x, y, z) = x + y + z$, where $+$ is a Boolean group operation;*

(V) *f is a semiprojection, i.e. there exists an index i $(1 \leq i \leq n)$ such that $f(x_1, \ldots, x_n) = x_i$ whenever the values of $x_1, \ldots, x_n$ are not pairwise distinct.*

The simplest examples of minimal clones of type (II), i.e. groupoids with a minimal clone, are semilattices and rectangular bands. We give the defining identities of some more groupoid varieties with a minimal clone in Table 1. (We have omitted the identity $xx = x$ everywhere, but of course these are all idempotent varieties.)

Affine spaces provide further examples of binary minimal clones. The clone of an affine space is minimal iff the base field is isomorphic to $\mathbb{Z}_p$ for some prime number p. In the following affine spaces are always meant to be affine spaces over $\mathbb{Z}_p$ (for an arbitrary prime p).

There are much less examples of minimal clones of type (III). The simplest ones are those containing just one nontrivial ternary operation. The clone generated by the median function $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ on any lattice is such a clone [PK]. Another example of a majority minimal clone is the clone generated by the dual discriminator function on any set [CsG, FP].

It seems to be a very hard problem to characterize minimal clones in full generality, but there are some results that describe minimal clones or minimal functions under certain assumptions.

One of the most natural approaches is to restrict the size of the underlying set of a concrete clone. E. Post determined all clones on the two-element set [Po]; seven of them are minimal. Minimal clones on the three-element set were described by B. Csákány [Cs2]; we quote the result for type (III) below. For the four-element set minimal clones of type (II) were described by B. Szczepara [Szcz]. We describe minimal majority functions on the four-element set in Theorem 6; the case of semiprojections remains open.

Theorem 2. [Cs2] *There are twelve minimal majority functions on the three-element set up to isomorphism, and they belong to three minimal clones containing* 1, 3 *and* 8 *majority operations respectively* (*see Table* 3).

Based on this theorem, B. Csákány obtained a characterization of minimal majority operations which are *conservative*, i.e. which preserve all subsets of the underlying set [Cs3].

Another possibility is to look for minimal functions satisfying certain identities. Probably the most natural result of this kind is the following characterization of semigroups with a minimal clone given by M. B. Szendrei ([SzM], see also [P^3]).

Theorem 3. [$\mathbf{P}^3$, SzM] *A semigroup with a minimal clone is either a left regular band, a right regular band or a rectangular band.*

Á. Szendrei and K. Kearnes investigated minimal clones generated by an operation that commutes with itself [KSz]. In the binary case this commutativity property is equivalent to the so-called entropic or medial law $(xy)(zu) = (xz)(yu)$, and the result is the following.

Theorem 4. [KSz] *Let $\mathbb{A}$ be an entropic groupoid with a minimal clone. Then $\mathbb{A}$ or its dual is an affine space, a rectangular band, a left normal band, a right semilattice or a p-cyclic groupoid for some prime p.*

Finally, let us quote a result of K. Kearnes describing abelian algebras with a minimal clone [Kea].

Theorem 5. **[Kea]** *If a minimal clone has a nontrivial abelian representation, then it is either unary, or the clone of an affine space, a rectangular band or a p-cyclic groupoid for some prime p.*

2. Minimal majority clones on the four-element set

Our goal in this chapter is to determine the minimal majority functions on the four-element set. This is a finite task, since it is possible to test in finitely many steps whether a function is minimal or not, and there are finitely many majority operations on a finite set. However, the four-element set is already very big from this point of view. There is only one majority operation on the two-element set, and $3^6 = 729$ on the three-element set, while on the four-element set we have $4^{24} = 281\,474\,976\,710\,656$ functions. Thus it seems hopeless to test them one by one, even with the help of a computer. After a long reduction process only three nonconservative functions remain up to isomorphism and permutation of variables that have a chance to be minimal. They turn out to be minimal; actually their clones are isomorphic to the three minimal majority clones on the three-element set. (Let us recall that the conservative case is settled in [Cs3].)

Theorem 6. **[Wa1]** *There are twelve nonconservative minimal majority functions on the four-element set up to isomorphism, and they belong to three minimal clones containing* 1, 3 *and* 8 *majority operations respectively* (*see Table* 4). *These three clones are isomorphic to the minimal majority clones of the three-element set.*

The number of minimal majority operations and clones is given in Table 5.

3. Minimal clones with few majority functions

In this chapter we describe minimal clones of type (III) with at most seven ternary operations (see [LP] for the analogous question in the binary case). A unique property of clones generated by a majority operation is that the minimality of such a clone depends only on its ternary functions. We denote the ternary part of $\mathcal{C}$ by $\mathcal{C}^{(3)}$, and we regard it as an algebra with one quaternary operation (the composition of ternary functions) and three constants (the projections).

First we prove a general theorem about the symmetries of the majority functions in a minimal clone which is an analogue of a theorem of J. Dudek and J. Gałuszka concerning minimal clones containing only commutative nontrivial binary operations [DG].

Theorem 7. **[Wa4]** *Let $\mathcal{C}$ be a majority minimal clone with finitely many ternary operations. If every nontrivial ternary operation in $\mathcal{C}$ is invariant under cyclic permutations of its variables, then $\mathcal{C}$ contains only one nontrivial ternary operation.*

The main result of this chapter describes minimal majority clones with at most four majority operations. It turns out that each such clone can be realized on the three-element set up to isomorphism of the ternary part of the clone.

Theorem 8. **[Wa4]** *If $\mathcal{C}$ is a majority minimal clone such that $\left|\mathcal{C}^{(3)}\right| \leq 7$, then $\mathcal{C}$ contains either one or three majority operations. In both cases $\mathcal{C}^{(3)}$ is uniquely determined up to isomorphism.*

Let us remark that the characterization is given up to isomorphism of $\mathcal{C}^{(3)}$, not $\mathcal{C}$ itself. In fact, there are infinitely many nonisomorphic minimal clones with one or three majority operations.

4. Minimal clones with weakly abelian representations

This chapter gives a generalization of Theorem 5 using a weaker term condition, called weak abelianness. It was proved in [Kea] that minimal clones of type (III) and (V) do not have nontrivial abelian representations, and the proof actually shows that they do not have nontrivial weakly abelian representations either. Every representation of a minimal clone of type (I) or (IV) is clearly abelian, therefore we only need to consider weakly abelian groupoids with a minimal clone.

First we show that if a distributive groupoid has a minimal clone, then it is entropic. Using this result we prove that every weakly abelian groupoid having a minimal clone is entropic. It is easy to check that nontrivial left (right) normal bands and nontrivial left (right) semilattices cannot be weakly abelian, therefore taking Theorem 4 into account, we get the same list of minimal clones as in Theorem 5.

Theorem 9. **[Wa2]** *If a minimal clone has a nontrivial weakly abelian representation, then it also has a nontrivial abelian representation. Therefore such a clone must be a unary clone, the clone of an affine space, a rectangular band or a p-cyclic groupoid for some prime p.*

Unary algebras, rectangular bands and affine spaces are all abelian, and it is not hard to show that every p-cyclic groupoid is weakly abelian. This fact yields an interesting homogeneity property for weakly abelian representations.

Theorem 10. **[Wa2]** *If a minimal clone has a nontrivial weakly abelian representation, then all representations are weakly abelian.*

We conclude with a theorem about rectangular and strongly abelian representations of minimal clones. A nontrivial affine space or p-cyclic groupoid cannot be rectangular, but unary algebras and rectangular bands are all strongly abelian. Thus these two term conditions are equivalent for groupoids with minimal clones.

Theorem 11. **[Wa2]** *If a minimal clone has a nontrivial rectangular representation, then it also has a nontrivial strongly abelian representation; moreover, all representations are strongly abelian. Such a clone must be unary or the clone of rectangular bands.*

5. Almost associative operations generating a minimal clone

In this chapter we generalize Theorem 3 by characterizing minimal clones generated by almost associative binary operations. To explain what we mean by this, we need a way to measure how far a binary operation is from being associative.

One way to measure associativity is to count the *nonassociative triples* in the groupoid. It is natural to say that the multiplication of $\mathbb{A}$ is almost associative if there is only one nonassociative triple, i.e. if $(ab)\,c = a\,(bc)$ fails for only one $(a, b, c) \in A^3$. These groupoids are called *Szász-Hájek groupoids* (SH-groupoids for short). The following theorem describes SH-groupoids with a minimal clone.

Theorem 12. **[Wa3]** *For any Szász-Hájek groupoid* $\mathbb{A}$ *the following two conditions are equivalent:*

(i) $\mathbb{A}$ *has a minimal clone;*

(ii) $\mathbb{A}$ *or its dual belongs to the variety* $\mathcal{B}$.

The elements of the unique nonassociative triple in an SH-groupoid generate an SH-groupoid whose proper subgroupoids are all semigroups. Such groupoids are called a *minimal SH-groupoids*. In [KT3-KT6] the project of characterizing minimal SH-groupoids was begun, but completed only for certain types. However, these types of groupoids do not have minimal clones (except for one groupoid), so the next theorem gives new minimal SH-groupoids.

Theorem 13. **[Wa3]** *Up to isomorphism and duality there are ten minimal SH-groupoid with a minimal clone.* (*Their multiplication tables can be found in Table* 7.)

Another way of measuring associativity is possible by considering the identities implied by associativity, and somehow counting how many of these are (not) satisfied. Let $s_{\mathbb{A}}\,(n)$ denote the number of those term functions of the groupoid

$\mathbb{A}$ which arise from the product $x_1 \cdot \ldots \cdot x_n$ by inserting parentheses in order to specify the order of the multiplications. The sequence $s_{\mathbb{A}}(1), s_{\mathbb{A}}(2), \ldots, s_{\mathbb{A}}(n), \ldots$ is called the *associative spectrum* of $\mathbb{A}$ [CsW].

Clearly, $s_{\mathbb{A}}(1) = s_{\mathbb{A}}(2) = 1$ for every groupoid $\mathbb{A}$, and $s_{\mathbb{A}}(3) = 1$ iff $\mathbb{A}$ is a semigroup. In the latter case $s_{\mathbb{A}}(n) = 1$ for all n by the general law of associativity. The smallest possible spectrum for a nonassociative multiplication is $1, 1, 2, 1, 1, \ldots$, so we could say that a binary operation is almost associative if its spectrum is this sequence. However, there is no groupoid having a minimal clone with this spectrum. Therefore we have to be more generous: in the following theorem we characterize groupoids with a minimal clone satisfying $s(4) < 5$. (The variety $\mathcal{A}$ in the theorem is defined by the identity $x(y(zu)) = x((yz)u)$).

Theorem 14. **[Wa3]** *For any groupoid $\mathbb{A}$ the following two conditions are equivalent:*

(i) *$\mathbb{A}$ has a minimal clone and $1 < s_{\mathbb{A}}(4) < 5$;*

(ii) *$\mathbb{A}$ is not a semigroup and $\mathbb{A}$ or its dual belongs to one of the varieties $\mathcal{B} \cap \mathcal{A}$, $\mathcal{C}_p$ or $\mathcal{D} \cap \mathcal{A}$ for some prime p.*

If these conditions are fulfilled, then we have $s_{\mathbb{A}}(n) = 2^{n-2}$ for $n \geq 2$.

Let us note finally that the class of groupoids found in Theorem 14 is disjoint from the class described in Theorem 12, i.e. there is no groupoid with a minimal clone that is almost associative in both the 'spectral' and the 'index' sense.

Összefoglaló

1. Bevezetés

Konkrét klónon egy adott halmazon értelmezett többváltozós függvények olyan összességét értjük, amely zárt az összetett függvények képzésére és tartalmazza a projekciókat. Az *absztrakt klónok* olyan heterogén algebrák, amelyek a konkrét klónbeli kompozícióműveletek struktúráját írják le. Egy $\mathcal{C}$ absztrakt klón *reprezentációja* olyan klónhomomorfizmus (illetve annak képe), ami $\mathcal{C}$-t valamely halmaz műveleteinek konkrét klónjába képezi le. A legfontosabb példákat klónokra algebrák termfüggvényeinek klónjai szolgáltatják.

Egy klónt akkor nevezünk *minimálisnak*, ha egyetlen valódi részklónja a triviális klón (a projekciókból álló klón). Egy nemtriviális klón akkor és csak akkor minimális, ha bármely nemtriviális eleme generálja. Természetes, hogy a lehető legkisebb változószámú generátort válasszuk. Ezeket a generátorokat *minimális függvényeknek* nevezzük. A minimális függvények öt típusba sorolhatók I. G. Rosenberg alábbi tétele szerint.

1. Tétel. [Ros] *Legyen f minimális aritású nemtriviális függvény egy minimális klónban. Ekkor f kielégíti az alábbi öt feltétel valamelyikét:*

(I) *f egyváltozós, és $f^2(x) = f(x)$ vagy $f^p(x) = x$ valamely p prímszámra;*

(II) *f idempotens kétváltozós művelet, azaz $f(x, x) = x$;*

(III) *f háromváltozós többségi függvény, azaz $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$;*

(IV) *$f(x, y, z) = x + y + z$, ahol $+$ egy elemi Abel 2-csoport művelete;*

(V) *f szemiprojekció, azaz létezik olyan i $(1 \leq i \leq n)$, hogy $f(x_1, \dots, x_n) = x_i$, ha az $x_1, \dots, x_n$ értékek között van ismétlődés.*

A legegyszerűbb példákat (II)-es típusú minimális klónokra, azaz minimális klónú grupoidokra, a félhálók és a derékszögű kötegek adják. Az 1. táblázatban megadjuk néhány további minimális klónnal rendelkező grupoidvarietás definiáló azonosságait. (Az $xx = x$ azonosságot nem írtuk ki sehol, de természetesen idempotens varietásokról van szó.)

Az affin terek további példákat szolgáltatnak binér minimális klónokra. Egy affin tér klónja akkor és csak akkor minimális, ha az alaptest izomorf a $\mathbb{Z}_p$ maradékosztálytesttel valamely p prímszámra. A továbbiakban affin téren mindig $\mathbb{Z}_p$ feletti affin teret értünk (tetszőleges p prímre).

Sokkal kevesebb példát ismerünk (III)-as típusú minimális klónra. A legegyszerűbbek azok, amelyek csak egy nemtriviális háromváltozós műveletet tartalmaznak. Tetszőleges hálón az $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ mediális függvény ilyen klónt generál [PK]. Egy másik példa többségi minimális klónra tetszőleges halmazon a duális diszkriminátor függvény által generált klón [CsG, FP].

A minimális klónok teljes általánosságban történő leírása nagyon nehéz problémának tűnik, vannak azonban olyan eredmények, amelyek bizonyos feltételek mellett karakterizálják a minimális klónokat.

A legtermészetesebb megközelítés az alaphalmaz méretének korlátozása. A kételemű halmazon E. Post meghatározta az összes klónt [Po], ezek közül hét minimális. Csákány Béla írta le a háromelemű halmaz minimális klónjait [Cs2], alább idézzük a (III)-as típusra vonatkozó tételt. A négyelemű halmazon B. Szczepara határozta meg a (II)-es típusú minimális klónokat [Szcz], a minimális többségi függvényeket pedig 6. Tételben adjuk meg. A négyelemű halmaz (V)-ös típusú minimális klónjainak leírása még nyitott probléma.

2. Tétel. **[Cs2]** *Izomorfia erejéig tizenkét minimális többségi függvény van a háromelemű halmazon, és ezek három minimális klónba tartoznak, amelyek rendre* 1, 3 *és* 8 *többségi függvényt tartalmaznak* (*lásd a* 3. *táblázatot*).

A fenti tétel segítségével Csákány Béla leírta a *konzervatív* minimális többségi függvényeket, vagyis azokat, amelyek megőrzik az alaphalmaz minden részhalmazát [Cs3].

Egy másik lehetséges megszorítás, hogy bizonyos azonosságokat kielégítő műveletek körében keressük a minimális függvényeket. Talán a legtermészetesebb ilyen eredmény a minimális klónú félcsoportok B. Szendrei Mária által adott jellemzése ([SzM], lásd még [P^3]).

3. Tétel. **[$\mathbf{P}^3$, SzM]** *A minimális klónú félcsoportok pontosan a bal- és jobbreguláris kötegek, valamint a derékszögű kötegek.*

Szendrei Ágnes és K. Kearnes vizsgálta azokat a minimális klónokat, amelyeket egy önmagával felcserélhető művelet generál [KSz]. Ez a felcserélhetőségi tulajdonság a kétváltozós esetben ekvivalens az $(xy)\,(zu) = (xz)\,(yu)$ entropikus, vagy mediális azonossággal.

4. Tétel. [KSz] *Legyen $\mathbb{A}$ egy minimális klónnal rendelkező entropikus grupoid. Ekkor $\mathbb{A}$ vagy duálisa affin tér, derékszögű köteg, balnormális köteg, jobbfélháló vagy p-ciklikus grupoid valamely p prímszámra.*

Végezetül idézzük K. Kearnes egy tételét, amely karakterizálja azokat az Abel-féle algebrákat, amelyek klónja minimális [Kea].

5. Tétel. [Kea] *Ha egy minimális klónnak létezik nemtriviális Abel-féle reprezentációja, akkor vagy egyváltozós, vagy pedig egy affin tér, egy derékszögű köteg vagy egy p-ciklikus grupoid klónja valamely p prímszámra.*

2. Többségi minimális klónok a négyelemű halmazon

Ezen fejezet célja a négyelemű halmaz minimális többségi függvényeinek meghatározása. Ez véges feladat, hiszen véges sok lépésben ellenőrizhető, hogy egy adott függvény minimális-e, és véges halmazon véges számú többségi függvény van. Mindazonáltal a négyelemű halmaz már meglehetősen nagy ebből a szempontból. A kételemű halmazon csak egy többségi függvény van, a háromeleműn pedig $3^6 = 729$, míg a négyelemű halmazon már $4^{24} = 281\,474\,976\,710\,656$ többségi függvény van. Ezért még számítógéppel is reménytelennek tűnik egyenként sorra venni az összes függvényt. Redukciós lépések hosszú sora után kiderül, hogy mindössze három olyan nemkonzervatív függvény marad, amelynek egyáltalán van esélye arra, hogy minimális legyen (izomorfia és a változók permutációja erejéig). Ezek valóban minimálisak, ugyanis klónjaik izomorfak a háromelemű halmaz minimális többségi klónjaival. (Emlékeztetünk rá, hogy a konzervatív minimális többségi függvények minden véges halmazon ismertek [Cs3].)

6. Tétel. [Wa1] *Izomorfia erejéig tizenkét nemkonzervatív minimális többségi függvény van a négyelemű halmazon, és ezek három minimális klónba tartoznak, amelyek rendre* 1, 3 *és* 8 *többségi függvényt tartalmaznak* (*lásd a* 4. *táblázatot*). *E három klón izomorf a háromelemű halmaz három többségi minimális klónjával.*

A minimális klónok és függvények számát az 5. táblázatban adjuk meg.

3. Kevés többségi függvényt tartalmazó minimális klónok

Ebben a fejezetben meghatározzuk a legfeljebb hét háromváltozós függvényt tartalmazó (III)-as típusú minimális klónokat (a kétváltozós esetre vonatkozó hasonló kérdést illetően lásd [LP]). A többségi függvény által generált klónok egy kivételes tulajdonsága, hogy a klón minimalitása csupán a benne található háromváltozós függvényeken múlik. A $\mathcal{C}$ klón háromváltozós részét $\mathcal{C}^{(3)}$ jelöli, ezt a halmazt egy négyváltozós művelettel (a háromváltozós függvények kompozíciója) és három konstanssal (a háromváltozós projekciók) ellátott algebrának tekintjük.

Először egy általános állítást bizonyítunk be a minimális klónokban található többségi függvények szimmetriáiról ami J. Dudek és J. Gałuszka csupa kommutatív nemtriviális kétváltozós műveletet tartalmazó minimális klónokról szóló tételének analogonja [DG].

7. Tétel. [**Wa4**] *Legyen $\mathcal{C}$ egy többségi minimális klón véges sok háromváltozós művelettel. Ha $\mathcal{C}$-ben minden nemtriviális háromváltozós művelet invariáns változóinak ciklikus permutációjára, akkor $\mathcal{C}$ csak egy nemtriviális háromváltozós műveletet tartalmaz.*

A fejezet fő eredménye a legfeljebb négy többségi függvényt tartalmazó többségi minimális klónok leírása. Kiderül, hogy a klón háromváltozós részének izomorfiája erejéig minden ilyen klón realizálható a háromelemű halmazon.

8. Tétel. [**Wa4**] *Ha a $\mathcal{C}$ többségi minimális klónra $\left|\mathcal{C}^{(3)}\right| \leq 7$ teljesül, akkor $\mathcal{C}$ vagy egy vagy három többségi függvényt tartalmaz. Mindkét esetben $\mathcal{C}^{(3)}$ izomorfia erejéig egyértelműen meghatározott.*

Figyeljük meg, hogy a jellemzés $\mathcal{C}^{(3)}$, nem pedig $\mathcal{C}$ izomorfiája erejéig van megadva. Valójában végtelen sok nemizomorf többségi minimális klón van, amely egy vagy három többségi függvényt tartalmaz.

4. Minimális klónok gyengén Abel-féle reprezentációi

Ezen fejezetben az 5. Tételt általánosítjuk egy gyengébb term-feltétel, a gyenge Abel-féleség használatával. K. Kearnes bizonyította be, hogy (III)-as és (V)-ös típusú minimális klónnak nem lehet nemtriviális Abel-féle reprezentátiója [Kea], és a bizonyítás valójában azt is mutatja, hogy gyengén Abel-féle reprezentációja sem lehet. Az (I)-es és (IV)-es típusú minimális klónoknak viszont minden reprezentációja Abel-féle, így elegendő a minimális klónnal rendelkező gyengén Abel-féle grupoidokat vizsgálnunk.

Először megmutatjuk, hogy minden minimális klónú disztributív grupoid entropikus. Ennek segítségével igazoljuk, hogy a minimális klónnal rendelkező gyengén Abel-féle grupoidok entropikusak. Könnyű ellenőrizni, hogy egy nemtriviális balnormális (jobbnormális) köteg illetve balfélháló (jobbfélháló) nem lehet gyengén Abel-féle, így a 4. Tételt figyelembe véve pontosan ugyanazokat a minimális klónokat kapjuk, mint az 5. Tételben.

9. Tétel. [**Wa2**] *Ha egy minimális klónnak van nemtriviális gyengén Abel-féle reprezentációja, akkor van nemtriviális Abel-féle reprezentációja is. Ezért egy ilyen* *klón csak unér lehet, vagy pedig egy affin tér, egy derékszögű köteg vagy egy p-ciklikus grupoid klónja valamely p prímszámra.*

Az unér algebrák, a derékszögű kötegek és az affin terek mindig Abel-félék, és nem nehéz megmutatni, hogy minden p-ciklikus grupoid gyengén Abel-féle. Ez a tény egy érdekes homogenitási tulajdonságot ad a gyengén Abel-féle reprezentációkra.

10. Tétel. [**Wa2**] *Ha egy minimális klónnak létezik nemtriviális gyengén Abel-féle reprezentációja, akkor minden reprezentációja gyengén Abel-féle.*

Végül minimális klónok derékszögű és erősen Abel-féle reprezentációiról mondunk ki egy tételt. Egy nemtriviális affin tér vagy p-ciklikus grupoid nem lehet derékszögű, viszont az unér algebrák és a derékszögű kötegek mind erősen Abel-félék. Tehát ez a két term-feltétel egybeesik a minimális klónú grupoidok körében.

11. Tétel. [**Wa2**] *Ha egy minimális klónnak van nemtriviális derékszögű reprezentációja, akkor van nemtriviális erősen Abel-féle reprezentációja is, sőt minden reprezentációja erősen Abel-féle. Egy ilyen klón csak unér lehet, vagy pedig egy derékszögű köteg klónja.*

5. Majdnem asszociatív műveletek által generált minimális klónok

A 3. Tétel két lehetséges általánosítását adjuk meg ebben a fejezetben a minimális klónt generáló majdnem asszociatív kétváltozós műveletek leírásával. Hogy ezt pontosabban meg tudjuk fogalmazni, mérnünk kell valahogyan, hogy egy adott művelet milyen messze van attól, hogy asszociatív legyen.

Egy lehetséges mód az asszociativitás mérésére, hogy meghatározzuk a *nemasszociatív hármasok* számát. Természetes azt mondani, hogy az $\mathbb{A}$ grupoid művelete majdnem asszociatív, ha csak egy nemasszociatív hármasa van, azaz $(ab)\,c = a\,(bc)$ teljesül egyetlen $(a,b,c) \in A^3$ kivételével. Az ilyen grupoidokat *Szász-Hájek grupoidoknak* nevezzük (röviden SH-grupoidok). A következő tételben jellemezzük a minimális klónú SH-grupoidokat.

12. Tétel. [**Wa3**] *Tetszőleges $\mathbb{A}$ Szász-Hájek grupoidra ekvivalens a következő két állítás:*

(i) $\mathbb{A}$ *klónja minimális;*

(ii) $\mathbb{A}$ *vagy duálisa a $\mathcal{B}$ varietásba tartozik.*

Egy SH-grupoid nemasszociatív hármasának elemei olyan SH-grupoidot generálnak, amelynek minden valódi részgrupoidja félcsoport. Az ilyen grupoidokat *minimális SH-grupoidoknak* nevezzük. A minimális SH-grupoidok szisztematikus leírásárát T. Kepka és M. Trch kezdte el, de a karakterizáció csak bizonyos típusú SH-grupoidok esetén teljes [KT3-KT6]. Egyetlen kivételtől eltekintve az ilyen típusú grupoidok klónja nem lehet minimális, így a következő tétel új minimális SH-grupoidokat szolgáltat.

13. Tétel. **[Wa3]** *Izomorfia és dualitás erejéig tíz minimális klónú minimális SH-grupoid létezik.* (*A művelettáblázataikat lásd a 7. táblázatban.*)

Egy másik módja az asszociativitás mérésének, hogy számba vesszük, hogy az asszociativitásból következő azonosságok közül mennyi (nem) teljesül. Jelölje $s_{\mathbb{A}}(n)$ az $\mathbb{A}$ grupoid azon termfüggvényeinek számát, amelyek úgy keletkeznek, hogy az $x_1, x_2, \ldots, x_n$ szorzatot zárójelekkel látjuk el, hogy a szorzások sorrendje egyértelműen meghatározott legyen. Az $s_{\mathbb{A}}(1), s_{\mathbb{A}}(2), \ldots$ sorozatot az $\mathbb{A}$ grupoid *asszociatív spektrumának* nevezzük [CsW].

Világos, hogy bármely $\mathbb{A}$ grupoidra $s_{\mathbb{A}}(1) = s_{\mathbb{A}}(2) = 1$, és $s_{\mathbb{A}}(3) = 1$ akkor és csak akkor, ha $\mathbb{A}$ félcsoport. Az utóbbi esetben az általános asszociativitás tétele szerint $s_{\mathbb{A}}(n) = 1$ teljesül minden n pozitív egész számra. A legkisebb spektrum tehát, ami a nemasszociatív műveletek körében felléphet, az $1, 1, 2, 1, 1, \ldots$ sorozat, ezért azokat a műveleteket nevezhetnénk majdnem asszociatívnak, amelyeknek a spektruma megegyezik ezzel a sorozattal. A minimális klónú grupoidok között azonban nem létezik olyan grupoid, amelynek ilyen kicsi lenne a spektruma. Ezért nagyvonalúbbnak kell lennünk: a 14. Tételben azokat a minimális klónt generáló kétváltozós műveleteket fogjuk meghatározni, amelyek spektrumára $s(4) < 5$ teljesül. (A tételben szereplő $\mathcal{A}$ varietást az $x(y(zu)) = x((yz)u)$ azonosság definiálja.)

14. Tétel. **[Wa3]** *Tetszőleges $\mathbb{A}$ grupoidra ekvivalens a következő két állítás:*

(i) *$\mathbb{A}$ klónja minimális, és $1 < s_{\mathbb{A}}(4) < 5$;*

(ii) *$\mathbb{A}$ nem félcsoport, és $\mathbb{A}$ vagy duálisa a $\mathcal{B} \cap \mathcal{A}$, $\mathcal{C}_p$ vagy $\mathcal{D} \cap \mathcal{A}$ varietások valamelyikébe tartozik* (*alkalmas p prímszámra*).

Ha ezen feltételek teljesülnek, akkor $s_{\mathbb{A}}(n) = 2^{n-2}$ minden $n \geq 2$ esetén.

A 14. Tételben és a 12. Tételben leírt grupoidok halmaza diszjunkt, tehát nem létezik olyan minimális klónú grupoid amely majdnem asszociatív „spektrális" és „indexes" értelemben is.

Tables and figures

$$
\begin{aligned}
\text{(semilattices) } \mathcal{SL}: &\ (xy)\,z = x\,(yz)\,, xy = yx \\
\text{(rectangular bands) } \mathcal{RB}: &\ (xy)\,z = x\,(yz)\,, xyz = xz \\
\text{(right normal bands) } \mathcal{RNB}: &\ (xy)\,z = x\,(yz)\,, xyz = yxz \\
\text{(right regular bands) } \mathcal{RRB}: &\ (xy)\,z = x\,(yz)\,, xyx = yx \\
\mathcal{B}: &\ x\,(yx) = (xy)\,x = (xy)\,y = (xy)\,(yx) = x\,(xy) = xy \\
\mathcal{D}: &\ x\,(yx) = (xy)\,x = (xy)\,y = (xy)\,(yx) = xy, \\
&\ x \cdot \overleftarrow{x \cdot y_1 \cdot \ldots \cdot y_n} = x \ \ (n = 1, 2, \ldots) \\
\mathcal{D} \cap \mathcal{A}: &\ x\,(yz) = xy, xy^2 = xy \\
\text{(right semilattices) } \mathcal{RSL}: &\ x\,(yz) = xy, xy^2 = xy, (xy)\,z = (xz)\,y \\
\text{(p-cyclic groupoids) } \mathcal{C}_p: &\ x\,(yz) = xy, xy^p = x,\ \ (xy)\,z = (xz)\,y
\end{aligned}
$$

Table 1: Some groupoid varieties with minimal clones

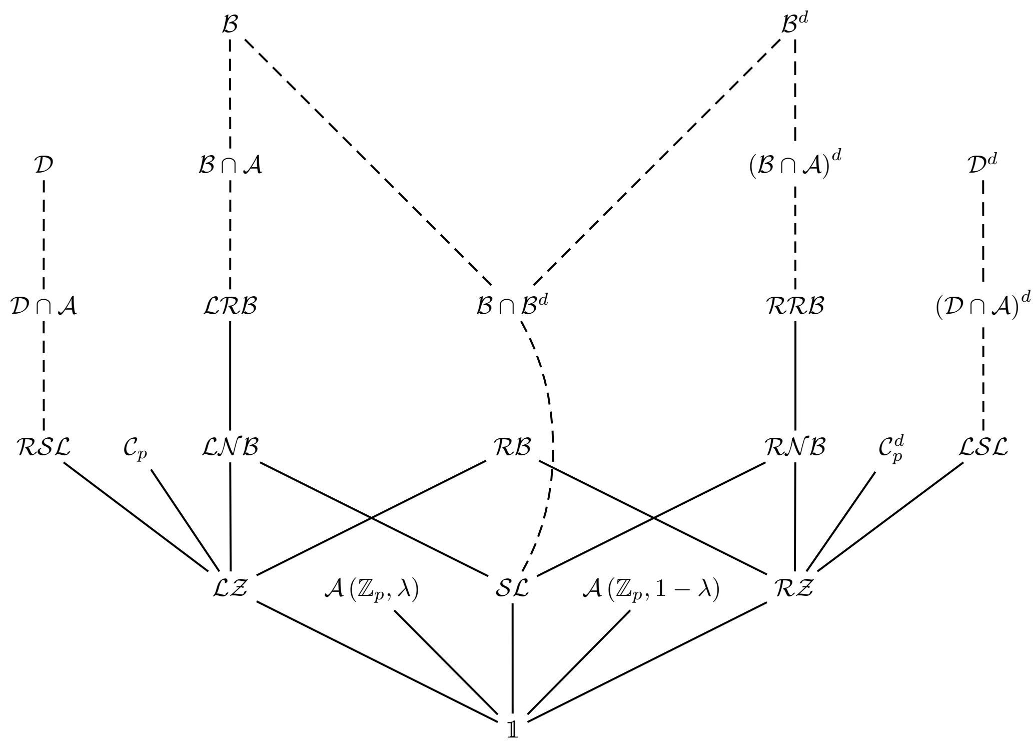


Figure 2: Some groupoid varieties with minimal clones

	m_1	m_2			m_3							
$(1,2,3)$	1	1	2	3	3	3	1	3	1	1	3	1
$(2,3,1)$	1	2	3	1	3	1	3	3	1	3	1	1
$(3,1,2)$	1	3	1	2	3	3	3	1	1	1	1	3
$(2,1,3)$	1	2	1	3	1	3	1	1	3	1	3	3
$(1,3,2)$	1	1	3	2	1	1	1	3	3	3	3	1
$(3,2,1)$	1	3	2	1	1	1	3	1	3	3	1	3

Table 3: Minimal majority functions on the 3-element set

	M_1	M_2			M_3							
$(1,2,3)$	4	4	2	3	3	3	4	3	4	4	3	4
$(2,3,1)$	4	2	3	4	3	4	3	3	4	3	4	4
$(3,1,2)$	4	3	4	2	3	3	3	4	4	4	4	3
$(2,1,3)$	4	2	4	3	4	3	4	4	3	4	3	3
$(1,3,2)$	4	4	3	2	4	4	4	3	3	3	3	4
$(3,2,1)$	4	3	2	4	4	4	3	4	3	3	4	3
$\{1,2,4\}$	4	4	4	4	4	4	4	4	4	4	4	4
$\{1,3,4\}$	4	4	4	4	4	4	4	4	4	4	4	4
$(4,2,3)$	4	4	2	3	3	3	4	3	4	4	3	4
$(2,3,4)$	4	2	3	4	3	4	3	3	4	3	4	4
$(3,4,2)$	4	3	4	2	3	3	3	4	4	4	4	3
$(2,4,3)$	4	2	4	3	4	3	4	4	3	4	3	3
$(4,3,2)$	4	4	3	2	4	4	4	3	3	3	3	4
$(3,2,4)$	4	3	2	4	4	4	3	4	3	3	4	3

Table 4: Nonconservative minimal majority functions on the 4-element set (The middle two rows mean that if $\{a,b,c\}$ equals $\{1,2,4\}$ or $\{1,3,4\}$, then the value of the functions on (a,b,c) is 4.)

	cons.	noncons.	all
minimal functions	32646	232	32 878
minimal functions up to isomorphism	1653	12	1665
minimal clones	2401	40	2441
minimal clones up to algebra isomorphism	126	3	129
minimal clones up to clone isomorphism	123	3	124

Table 5: The number of minimal majority functions and clones on the 4-element set

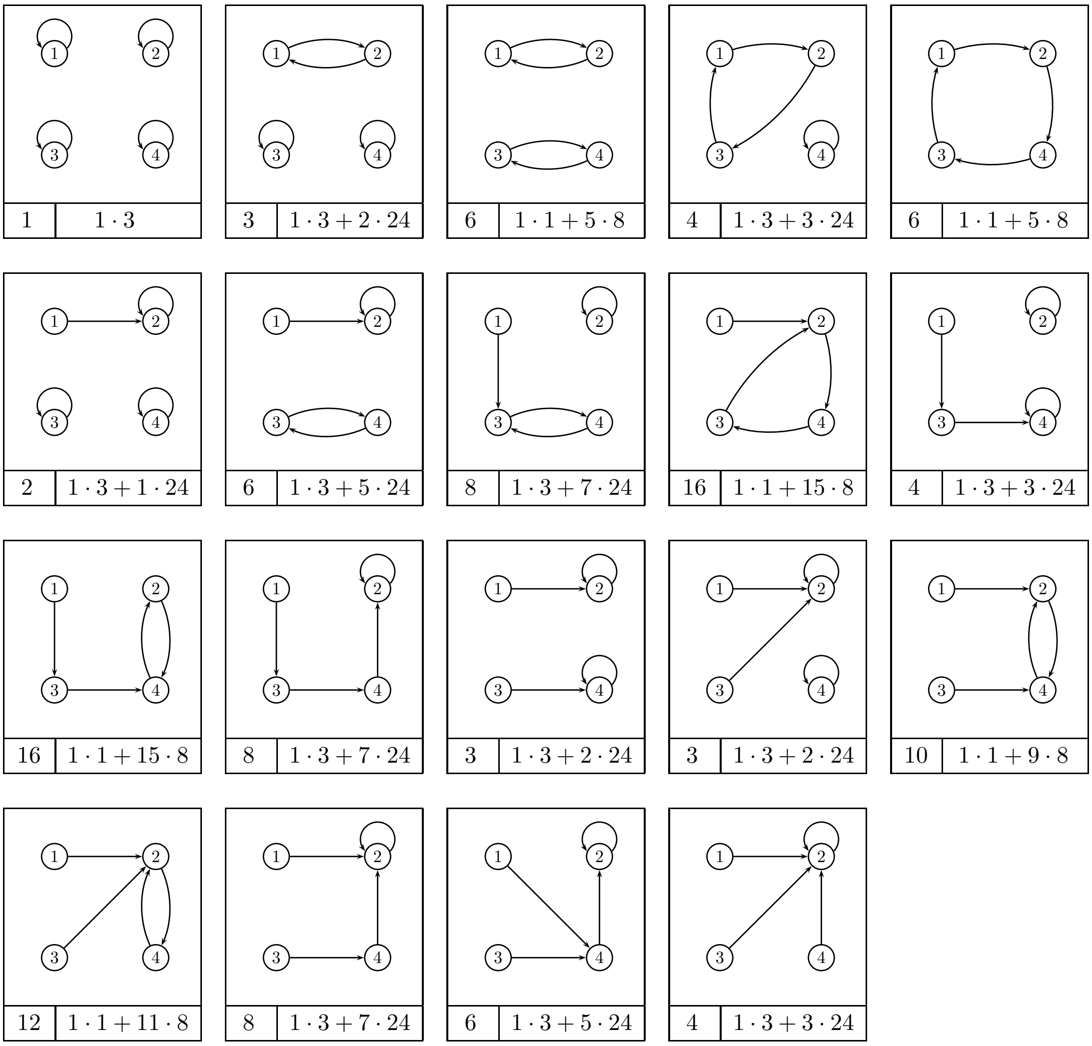


Figure 6: Isomorphism classes of minimal majority functions on the 4-element set

$\mathbb{G}_1$	a	b	c	e	f
a	a	a	c	f	f
b	b	b	e	e	e
c	c	c	c	c	c
e	e	e	e	e	e
f	f	f	f	f	f

$\mathbb{G}_2$	a	b	c	e
a	a	a	c	e
b	b	b	e	e
c	c	c	c	c
e	e	e	e	e

$\mathbb{G}_3$	a	b	c
a	a	a	c
b	b	b	b
c	c	c	c

$\mathbb{G}_4$	a	b	c	e	f	g
a	a	a	g	f	f	g
b	a	b	e	e	f	g
c	c	c	c	c	c	c
e	e	e	e	e	e	e
f	f	f	f	f	f	f
g	g	g	g	g	g	g

$\mathbb{G}_5$	a	b	c	e	g
a	a	a	g	e	g
b	a	b	e	e	g
c	c	c	c	c	c
e	e	e	e	e	e
g	g	g	g	g	g

$\mathbb{G}_6$	a	b	c	d	f	g	h	i
a	a	d	f	d	f	g	d	g
b	h	b	c	h	i	i	h	i
c	c	c	c	c	c	c	c	c
d	d	d	g	d	g	g	d	g
f	f	f	f	f	f	f	f	f
g	g	g	g	g	g	g	g	g
h	h	h	i	h	i	i	h	i
i	i	i	i	i	i	i	i	i

$\mathbb{G}_7$	a	b	c	d	f	g	h
a	a	d	f	d	f	g	d
b	h	b	c	h	g	g	h
c	c	c	c	c	c	c	c
d	d	d	g	d	g	g	d
f	f	f	f	f	f	f	f
g	g	g	g	g	g	g	g
h	h	h	g	h	g	g	h

$\mathbb{G}_8$	a	b	c	d	f	g
a	a	d	f	d	f	g
b	d	b	c	d	g	g
c	c	c	c	c	c	c
d	d	d	g	d	g	g
f	f	f	f	f	f	f
g	g	g	g	g	g	g

$\mathbb{G}_9$	a	b	c	d	f	h
a	a	d	f	d	f	d
b	h	b	c	h	h	h
c	c	c	c	c	c	c
d	d	d	d	d	d	d
f	f	f	f	f	f	f
h	h	h	h	h	h	h

$\mathbb{G}_{10}$	a	b	c	d	f
a	a	d	f	d	f
b	d	b	c	d	d
c	c	c	c	c	c
d	d	d	d	d	d
f	f	f	f	f	f

Table 7: Minimal Szász-Hájek groupoids with a minimal clone

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