

Approximation of convex bodies by polytopes

Outline of Ph.D. thesis

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2010

Szeged

1 Introduction

The research problems considered in the thesis originate from the area of polytopal approximation of convex bodies. The results fall into two broad categories, one is the best approximation of convex bodies by polytopes, the other is approximation of convex bodies by random polytopes.

The dissertation is based on the following papers of the author.

- I. Bárány, F. Fodor, V. Vígh: Intrinsic volumes of inscribed random polytopes in smooth convex bodies, *Adv. Appl. Probab.* (2009), 1–17, submitted for publication, available at arXiv:0906.0309v1.
- K. J. Böröczky, F. Fodor, M. Reitzner, V. Vígh: Mean width of random polytopes in a reasonable smooth convex body, *J. Multivariate Anal.*, **100** (2009), 2287–2295.
- K. J. Böröczky, F. Fodor, V. Vígh: Approximating 3-dimensional convex bodies by polytopes with a restricted number of edges, *Beiträge Algebra Geom.*, **49** (2008), no. 1, 177–193.
- V. Vígh: Typical faces of best approximating polytopes with a restricted number of edges, *Acta Sci. Math. (Szeged)*, **75** (2009), no. 1-2, 313–327.

In this outline we use the same numbering and labeling as in the thesis.

2 Best approximation of convex bodies by polytopes

Let K be a convex body in \mathbb{E}^d and let $0 \leq k \leq d - 1$ be an integer. One of the most often studied questions is how well one can approxi-

mate K with polytopes that have a restricted number of k -faces. These problems have become well understood in the last 30 years in the case if $k = 0$ or $k = d - 1$, that is, when the number of vertices or facets is restricted. Almost all results are asymptotic in nature, they are mainly due to R. Schneider, P. M. Gruber, M. Ludwig and K. J. Böröczky. There is a lack of results for the case when the number of intermediate dimensional faces is prescribed. In 2000 K. J. Böröczky [17] partially solved these problems, he gave upper and lower estimates of matching order of magnitude. Precise asymptotic formulas were not known till very recently. In Theorem 2.2.1 we solved the first interesting case, when $d = 3$ and $k = 1$. We measure the distance between convex bodies with the Hausdorff-metric. The analogous statement for volume approximation was proved by K. J. Böröczky, S. S. Gomez and P. Tick [22].

A more precise formulation of the problem is as follows. Let K be a 3-dimensional convex body with C^2 smooth boundary and let \mathcal{P}_n^c be the set of 3-polytopes with at most n edges that contain K , similiary, let \mathcal{P}_n^i be the set of 3-polytopes with at most n edges contained in K :

$$\mathcal{P}_n^c := \{P \mid P \supset K \text{ is a polytope with at most } n \text{ edges}\},$$

$$\mathcal{P}_n^i := \{P \mid P \subset K \text{ is a polytope with at most } n \text{ edges}\}.$$

There exist (not necessarily unique in general) polytopes $P_n^c \in \mathcal{P}_n^c$ and $P_n^i \in \mathcal{P}_n^i$ such that

$$\delta_H(P_n^c, K) = \inf_{P \in \mathcal{P}_n^c} \delta_H(P, K) \quad \text{and} \quad \delta_H(P_n^i, K) = \inf_{P \in \mathcal{P}_n^i} \delta_H(P, K),$$

that is their Hausdorff distances $\delta_H(P_n^c, K)$ and $\delta_H(P_n^i, K)$ from K are minimal. The first major result of Chapter 2 of the thesis is Theorem 2.2.1.

Theorem 2.2.1 (page 13, [21] Böröczky, Fodor, Vígh)

$$\delta_H(K, P_n^c), \delta_H(K, P_n^i) \sim \frac{1}{2} \int_{\partial K} \kappa^{1/2}(x) dx \cdot \frac{1}{n}, \quad \text{as } n \rightarrow \infty. \quad (1)$$

Here $\kappa(x)$ denotes the Gauss-curvature of ∂K at x and we integrate with respect to the 2-dimensional Hausdorff-measure on ∂K .

The following natural question arises here following the work of Gruber [36], [37], and Böröczky, Tick and Wintsche [24]. Can we say something more about the geometry of the best approximating polytopes? The answer is yes, we can determine the approximate shape and size of almost all of its faces. The second major result of Chapter 2 is Theorem 2.2.2.

Theorem 2.2.2 (page 14, [72] Vígh)

The typical faces of both P_n^i and P_n^c are squares with respect to the density function $\kappa^{1/2}(x)$ as $n \rightarrow \infty$.

The meaning of this theorem is the following. Let F be a face of P_n and $x_F \in \partial K$ a point where the outer normal is also a normal of the affine hull of F . Almost every face F of P_n is such a quadrilateral that is very close to a square with respect to the second fundamental form of ∂K at x_f , and that has area

$$\frac{\int_{\partial K} \kappa^{1/2}(x) dx}{f(n) \kappa^{1/2}(x_F)},$$

where $f(n)$ stands for the number of the faces of P_n .

The proof of Theorem 2.2.1 consists of two parts, we established matching upper and lower bounds on $\delta_H(K, P_n^c)$. In both parts the main idea was to divide the boundary of K into small enough pieces, and over each piece we used the osculating paraboloid of the surface to approximate ∂K locally. In the course of the proof of the upper bound, we constructed a polyhedral surface with a prescribed number of edges,

which approximates ∂K well. To obtain the lower bound we applied various algebraic and geometric inequalities. To prove Theorem 2.2.2 we needed the stability version of the inequalities we used to obtain the lower bound in (1).

The heart of the proofs is the following lemma, which resembles to the famous Momentum Theorem of L. Fejes Tóth [28].

Lemma 2.5.5 (page 19, [21] Böröczky, Fodor, Vígh and [72] Vígh)
Let $q(x)$ be a positive definite quadratic form on \mathbb{R}^2 and $\alpha \leq 0$ a real number. Let $G = [p_1, p_2, \dots, p_k]$ be a k -gon with vertices $\{p_i\}$. Then

$$\max_{x \in G} (q(x) - \alpha) \geq \frac{2}{k} \cdot A(G) \sqrt{\det q}. \quad (2)$$

Furthermore, if $k \neq 4$, then

$$\max_{x \in G} (q(x) - \alpha) > 1.04 \cdot \frac{2}{k} \cdot A(G) \sqrt{\det q}. \quad (3)$$

If

$$\max_{x \in G} (q(x) - \alpha) \leq (1 + \varepsilon) \cdot \frac{2}{k} \cdot A(G) \sqrt{\det q}, \quad (4)$$

then G is $O(\sqrt[4]{\varepsilon})$ -close to a q -square.

3 Random polytopes

In Chapter 3 we consider another aspect of polytopal approximation of convex bodies, that is we consider random polytopes. The most widely used model is the following. Let K be a convex body in E^d with volume 1, so the uniform probability measure and the Lebesgue-measure coincide in K . Choose n random points x_1, x_2, \dots, x_n from K independently and according to the uniform distribution. The convex hull $\text{conv}(x_1, \dots, x_n)$ of these points is called a random polytope in K , and we denote it by K_n . One of the central problems in stochastic

geometry is to understand the behavior of K_n . The main goals are to obtain information on the distribution of key geometric functionals of K_n .

It is clear, that the behavior of K_n strongly depends on the boundary structure of the mother body K , which implies, that the cases when K is a polytope or K has smooth boundary are quite different. For the case when ∂K is C_+^3 , and hence $\kappa(x) > 0$ for all $x \in \partial K$, R. Schneider, J.A. Wieacker [66] proved that

$$W(K) - \mathbb{E}W(K_n) \sim \frac{2\Gamma(\frac{2}{d+1})}{d(d+1)^{\frac{d-1}{d+1}}\kappa_d\kappa_{d-1}^{\frac{2}{d+1}}} \int_{\partial K} \kappa(x)^{\frac{d+2}{d+1}} dx \cdot \frac{1}{n^{\frac{2}{d+1}}}, \quad (5)$$

where $W(\cdot)$ denotes the mean width, κ_d is the volume of the Euclidean d -dimensional unit ball and $\mathbb{E}(\cdot)$ is the expectation. Recently, the smoothness condition was relaxed to C_+^2 by M. Reitzner [53].

Our first goal is to prove a further generalization of (5). We say that a convex body K has a rolling ball if there exists a $\rho > 0$ such that any $x \in \partial K$ lies in some ball of radius ρ contained in K . According to D. Hug [41], the existence of a rolling ball is equivalent saying that the exterior unit normal at $x \in \partial K$ is a Lipschitz function of x . The first major result of Chapter 3 extends (5) in the following way.

Theorem 3.1.2 (page 45, [20] Böröczky, Fodor, Reitzner, Vígh)

The asymptotic formula (5) holds for any convex body K of volume one which has a rolling ball.

Furhermore, Example 3.1.3 on page 45 states that there exists a K which has C_+^∞ boundary except at one point where it is only C^1 and (5) does not hold for K . This shows that Theorem 3.1.2 is essentially optimal.

Example 3.1.3 (page 45, [20] Böröczky, Fodor, Reitzner, Vígh) *If K is a convex body in \mathbb{R}^d such that $o \in \partial K$, ∂K is C_+^∞ on $\partial K \setminus o$, and the*

graph of $f(x) = \|x\|^{\frac{3d+1}{3d}}$ on $\mathbb{R}^{d-1} \cap B^d$ is part of ∂K then $\mathbb{E}(W(K) - W(K_n)) \geq \gamma n^{\frac{-4d}{3d^2+1}}$ where $\gamma > 0$ depends on d and $\frac{4d}{3d^2+1} < \frac{2}{d+1}$.

Asymptotic upper and lower bounds for the variance are needed to prove the strong law of large numbers and central limit theorems, see [12] and [13]. As a second major result of Chapter 3, we estimate the variance of all intrinsic volumes of K_n , if the body K has a C_+^2 smooth boundary.

Theorem 3.1.5 (page 48, [10] Bárány, Fodor, Vígh)

Let K be a convex body in \mathbb{E}^d with a C_+^2 smooth boundary. For all $s = 1, \dots, d$ there exist positive constants γ_1 and γ_2 depending only on d, s and K such that

$$\gamma_1 n^{-\frac{d+3}{d+1}} \leq \text{Var } V_s(K_n) \leq \gamma_2 n^{-\frac{d+3}{d+1}} \quad (6)$$

as $n \rightarrow \infty$, where $V_s(\cdot)$ stands for the s th intrinsic volume.

In addition, in the case of mean width we relaxed the smoothness condition on K , similarly to Theorem 3.1.2.

Theorem 3.1.6 (page 48, [20] Böröczky, Fodor, Reitzner, Vígh)

If K is a d -dimensional convex body of volume one with a rolling ball then

$$\gamma_1 n^{-\frac{d+3}{d+1}} < \text{Var} W(K_n) < \gamma_2 n^{-\frac{d+3}{d+1}},$$

where the positive constants γ_1, γ_2 depend on K and d .

We note that for Theorem 3.1.5 we gave a detailed proof only if K is the unit ball, and only sketched the proof for the general case. The reason for this is that the proof of the C_+^2 case is essentially the same as the case of the ball except some minor technical details. The proofs of the lower bounds in Theorem 3.1.5 and in Theorem 3.1.6 are very

similar, hence we gave a proof only for Theorem 3.1.5. The main idea of the proof of the lower bound is that we define small independent caps, and we show that the variance is “large” in each cap. From the properties of the variance the required estimate follows.

The proofs of the upper bounds in Theorem 3.1.5 and Theorem 3.1.6 are, however, completely different. To obtain the upper bound in Theorem 3.1.6 we applied integral geometric tools. In the case of Theorem 3.1.5 the key idea is to use the Economical Cap Covering Theorem of Bárány and Larman [11].

If K is a convex body, then a cap of K is a set $C = K \cap H_+$, where H_+ is closed half-space. We define the function $v : K \rightarrow \mathbb{R}$ as

$$v(x) := \min\{\lambda_d(K \cap H_+) \mid x \in H_+ \text{ and } H_+ \text{ is a closed half-space}\}.$$

The set $K(t) = K(v \leq t) = \{x \in K \mid v(x) \leq t\}$ is called the wet part of K with parameter $t > 0$.

Economical Cap Covering Theorem ([11] Bárány, Larman)

Assume that K is a convex body with unit volume, and $0 < t < t_0 = (2d)^{-2d}$. Then there are caps C_1, \dots, C_m and pairwise disjoint convex sets C'_1, \dots, C'_m such that $C'_i \subset C_i$ for each i , and

$$(i) \bigcup_1^m C'_i \subset K(t) \subset \bigcup_1^m C_i,$$

$$(ii) V_d(C'_i) \gg t \text{ and } V_d(C_i) \ll t \text{ for each } i,$$

(iii) for each cap C with $C \cap K(v > t) = \emptyset$ there is a C_i containing C .

The upper bound in (6) combined with the main results of [3] and [53] implies the strong law of large numbers by standard arguments, as it is stated in Theorem 3.1.7.

Theorem 3.1.7 (page 49, [10] Bárány, Fodor, Vígh)

If K is a convex body with C_+^2 boundary and K_n is the random polytope

inscribed in K , then

$$\lim_{n \rightarrow \infty} (V_s(K) - V_s(K_n)) \cdot n^{\frac{2}{d+1}} = c_{d,j} \cdot \kappa_d^{\frac{2}{d+1}} \int_S (\tau_{d-1}(x))^{\frac{1}{d+1}} \tau_{d-j}(x) dx.$$

with probability 1.

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