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Topological lower bounds for multichromatic numbers

Abstract of the Ph.D. Thesis

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Introduction

In the early 1970's Gilbert [18] introduced s -tuple colorings of graphs motivated by practical problems. Saul Stahl studied the properties of this colorings in [32]. In that paper he formulated the conjecture on the multichromatic number of the Kneser graphs. The first topological lower bound was given by László Lovász, when he settled the famous Kneser conjecture in [22]. Lovász's method produced further lower bounds for the chromatic number (see [35,24,3]). Motivated by Stahl's conjecture we generalize these bounds for the multichromatic number of G in [12,29], and we apply these bounds for the Kneser graph $KG_{m,n}$ in [28].

1. Methods of Combinatorial Topology

The topic of the dissertation is ranging over three fields. We apply the tools of graph theory, algebraic topology and combinatorics. The most important tools of our investigations are modern technics of combinatorial topology such as Discrete Morse Theory and Nerve Theorem.

2. Topological lower bounds for the chromatic number

The general idea for obtaining a topological lower bound for the chromatic number of a graph G is first associating a graph complex $\mathcal{K}(G)$ to G and then bound the chromatic number of G by a certain topological invariant of the complex $\mathcal{K}(G)$. This is summarized by the following scheme.

$$\begin{array}{ccc} \text{Graph } G & \longrightarrow & \text{Graph complex } \mathcal{K}(G) \\ & & \downarrow \\ \text{Lower bound for } \chi(G) & \longleftarrow & \text{Topological invariant of } \mathcal{K}(G) \end{array}$$

We study various graph complexes: the neighborhood complex $\mathcal{NK}(G)$ (introduced by L. Lovász [22]), the Lovász complex $\mathcal{LK}(G)$ (defined by J.W. Walker [35]) and the graph homomorphism complex $\mathcal{Hom}(H, G)$ (constructed by Babson and Kozlov [2]).

In 1978, László Lovász settled the famous Kneser conjecture [22], that the Kneser graph $KG_{m,n}$ is not $(m - 2n + 1)$ -colorable. In the proof he introduced a simplicial

complex related to a graph G : the *neighborhood complex* $\mathcal{NK}(G)$ is the simplicial complex whose vertices are the vertices of G and whose simplices are those subsets of $V(G)$ which have a common neighbor. He proved that the topological connectivity of this complex gives a lower bound for the chromatic number of the graph G .

Theorem 15. (Lovász [22]) *If the neighborhood complex of a graph G is l -connected, then $\chi(G) \geq l + 3$.*

Let $cn_G : 2^{V(G)} \rightarrow 2^{V(G)}$ be the common neighborhood map, introduced by Lovász ([22])

$$cn_G(A) = \{v \in V(G) : (v, a) \in E(G) \text{ for all } a \in A\}.$$

We say that a set $A \subset V(G)$ is *closed* if $cn_G \circ cn_G(A) = A$. The *Lovász complex* $\mathcal{LK}(G)$ is defined to be the subcomplex of the barycentric subdivision of $\mathcal{NK}(G)$ induced by the closed proper nonempty subsets of $V(G)$. J. W. Walker gave a topological lower bound on the chromatic numbers of the graph G by studying the Lovász complex $\mathcal{LK}(KG_{m,n})$.

Theorem 21. (Walker [35]) *Let G be a graph, then*

$$\chi(G) \geq \text{ind}(\mathcal{LK}(G)) + 2.$$

In [2] E. Babson and D. N. Kozlov defined the *homomorphisms complex* $\mathcal{Hom}(H, G)$. The motivation for considering $\mathcal{Hom}(H, G)$ came from the fact that $\mathcal{Hom}(K_2, G)$ is homotopy equivalent to the neighborhood complex $\mathcal{NK}(G)$, which played the central role in Lovász's proof of the Kneser Conjecture in [22]. The homotopy type of $\mathcal{Hom}(H, G)$ was determined in only very few special cases, for example, it was computed for the complex $\mathcal{Hom}(K_l, K_m)$.

Theorem 22. (Babson and Kozlov [2]) *For any positive integers $l \leq m$, $\mathcal{Hom}(K_l, K_m)$ is homotopy equivalent to a wedge of $(m - l)$ -dimensional spheres.*

As a corollary of their computations, Babson and Kozlov gave a generalization of Lovász's topological lower bound for the chromatic number. The index version of this bound:

Theorem 23. (Babson and Kozlov [2]) *Let G be a graph and let l be a positive integer, then*

$$\chi(G) \geq \text{ind}(\mathcal{Hom}(K_l, G)) + l.$$

3. s -tuple colorings

The idea of s -tuple colorings was introduced by Gilbert [18] in connection with the mobile radio frequency assignment problem. Other applications of s -tuple colorings include fleet maintenance, task assignment, and traffic phasing discussed in [27]. The graph-theoretical formulation of these problems is the following: There is a graph G . Make an assignment on G which assigns a set of s colors to each vertex of G so that the sets of colors assigned to adjacent vertices are disjoint.

In the early 1970's S. Stahl formulated the following conjecture on the multichromatic number of the Kneser graph in [32].

Conjecture 30. (Stahl [32]) *If $s = qn - r$ where $0 < q$ and $0 \leq r < n$, then*

$$\chi_s(KG_{m,n}) = qm - 2r.$$

The case $s = 1$ is the famous Kneser conjecture, which was settled by Lovász in [22]. Further, Stahl confirmed his conjecture for the case $s = qn$ in [32] by showing that $\chi_{qn}(KG_{m,n}) = qm$ for any positive integers n, m and q where $2n \leq m$. He also proved that $\chi_{s+1} \geq \chi_s + 2$ for any positive integer s . Since $\chi_n(KG_{m,n}) = m$, this yields

$$\chi_s(KG_{m,n}) = \chi_{s-1}(KG_{m,n}) + 2 \quad \text{for } 1 < s \leq n.$$

In general, the inequality

$$\chi_{qn+1}(KG_{m,n}) \geq \chi_{qn}(KG_{m,n}) + \chi_1(KG_{m,n})$$

would imply the conjecture.

The multichromatic numbers of the Kneser graph $KG_{m,n}$ are trivial for $n = 1$ and S. Stahl computed them for $n = 2$, $n = 3$ and $m = 2n + 1$ in [32] and [33]. Moreover, he proved the following inequality in [33].

Theorem 31. (Stahl [33]) *For any positive integers $2n \leq m$, and s*

$$qm - 2r - (n^2 - 3n + 4) \leq \chi_s(KG_{m,n}) \leq qm - 2r,$$

where $s = qn - r$, $0 \leq r < n$ and $0 < q$.

This proposition gives that $\chi_{qn}(KG_{m,n}) + \chi_1(KG_{m,n}) - f(n) \leq \chi_{qn+1}(KG_{m,n})$, where $f(n) = n^2 - 3n + 4$. Since f doesn't depend on m , for a fixed n and $c \in (0, 1)$ we have $\chi_{qn}(KG_{m,n}) + c\chi_1(KG_{m,n}) \leq \chi_{qn+1}(KG_{m,n})$ for m large enough. But if $m \leq n^2 - n + 4$, then Stahl's result implies only that $\chi_{qn}(KG_{m,n}) + 2 \leq \chi_{qn+1}(KG_{m,n})$.

4. Topological lower bounds for multichromatic numbers

In our main results we give on topological lower bounds for multichromatic numbers and we apply these bounds for the Kneser graph.

We use two methods for obtaining lower bounds on multichromatic numbers. First, we generalize the lower bounds of Walker, Babson and Kozlov to the multichromatic number of G . An s -tuple coloring of G with t colors is a $\gamma : G \rightarrow KG_{t,s}$ graph homomorphism. This induces a \mathbb{Z}_2 -map $c : |\mathcal{K}(G)| \rightarrow |\mathcal{K}(KG_{t,s})|$ for all \mathbb{Z}_2 -graph complexes $\mathcal{K}(\cdot)$. By computing the homotopy type of the complex $\mathcal{K}(KG_{t,s})$ we obtain topological lower bounds for the multichromatic number of G . We apply this idea first to the Lovász complex then to the graph homomorphism complex.

We have seen that the Lovász complex $\mathcal{LK}(KG_{t,s})$ is homotopy equivalent to a wedge of $(t - 2s)$ -dimensional spheres. Using this fact, it is easy to deduce the generalization of the Walker's Theorem for the multichromatic number.

Theorem 33. (Osztényi) *Let G be a graph and s positive integer, then*

$$\chi_s(G) \geq \text{ind}(\mathcal{LK}(G)) + 2s.$$

After obtaining this lower bound we determine the homotopy type of the graph homomorphism complex $\mathcal{H}om(K_n, KG_{t,s})$ as a generalization of the Theorem of Babson and Kozlov.

Theorem 34. (Osztényi [29]) *For any positive integers n, l and m with $\max\{2n, ln\} \leq m$, $\mathcal{H}om(K_l, KG_{m,n})$ is homotopy equivalent to a wedge of $(m - ln)$ -dimensional spheres.*

Using this theorem, we give a topological lower bound on the multichromatic number.

Theorem 36. (Osztényi [29]) *Let G be a graph and l, n positive integers, then $\chi_n(G) \geq \text{ind}(\mathcal{H}om(K_l, G)) + ln$.*

Next we study the neighborhood complex $\mathcal{NK}(G[K_s])$ and graph homomorphism complex $\mathcal{H}om(K_2, G[K_s])$ of the lexicographical product $G[K_s]$ for obtaining lower bounds on multichromatic numbers of a graph G . An s -tuple coloring of G is equivalent to an ordinary coloring of the lexicographical product $G[K_s]$. Therefore, the topological lower bounds for the ordinary chromatic number of $G[K_s]$ give another lower bound for the multichromatic number of G . First, Lovász's Theorem yields the following lower bound for the multichromatic number of a graph G :

$$\chi_s(G) \geq \text{conn}(\mathcal{NK}(G[K_s])) + 3.$$

Investigating the neighborhood complex $\mathcal{NK}(G[K_s])$ we find a connection between the topological connectivity of $\mathcal{NK}(G[K_s])$ and the topological connectivity of the so-called extended neighborhood complex of G described by the following theorem.

Theorem 41. (Osztényi [12]) *For any positive integers $s > l \geq 2$ and any graph G , the simplicial complex $\mathcal{NK}(G[K_s])$ is l -connected if and only if the extended neighborhood complex $\mathcal{EN}(G)$ is l -connected.*

The theorem implies that if $\text{conn}(\mathcal{EN}(G))$ is finite, then $\text{conn}(N(G[K_m])) = \text{conn}(\mathcal{EN}(G))$ for all $m \geq \text{conn}(\mathcal{EN}(G)) + 2$. Consequently, in this case the gaps between the multichromatic number and this lower bound can be arbitrarily large.

Further, the graph homomorphism complex $\mathcal{H}om(K_2, G[K_s])$ provides us a connection between the \mathbb{Z}_2 -index of $\mathcal{H}om(K_2, G[K_s])$ and the size of the largest clique (complete subgraph) of G described by Theorem 42.

Theorem 42. (Csorba [12]) *Let G be a graph and $s \geq |V(G)|$ positive integer, then*

$$\text{ind}(\mathcal{H}om(K_2, G[K_s])) + 2 = s \cdot \omega(G).$$

The great advantage of this result is that the expression $\text{ind}(\mathcal{H}om(K_2, G[K_s])) + 2$ is exactly the same as the lower bound for the multichromatic number asserted by the Theorem of Babson and Kozlov. This can be summarized as the following trivial lower bound:

Theorem 43. (Csorba [12]) *Let G be a graph and $s \geq |V(G)|$ positive integer, then*

$$\chi_s(G) \geq s \cdot \omega(G).$$

5. The study of the Stahl's conjecture

The generalization of Walker's and Babson-Kozlov's theorems seems to be a good idea, however this method does not lead to sharper bounds for the multichromatic number of $KG_{m,n}$ than those already known. On the other hand we show in the chapter that the topological connectivity number of the complex $\mathcal{EN}(KG_{m,n})$ is finite, thus applying Lovász's theorem to the lexicographical product $KG_{m,n}[K_s]$ remains also fruitless in proving Stahl's conjecture. Another trial for proving the conjecture is using Csorba's result (Theorem 43). Unfortunately this does not improve the previously obtained lower bound. These subsequent unsuccessful trials indicate that the applied topological methods alone are not sufficient for dealing with the problem. To overcome this difficulty we use the Lovász poset. An s -tuple coloring of the Kneser graph $KG_{m,n}$ with t color is a graph homomorphism $KG_{m,n} \rightarrow KG_{t,s}$, which induces an orthomap between the Lovász posets $LP(KG_{m,n}) \rightarrow LP(KG_{t,s})$. The advantage of this poset method is that this orthomap is sensitive for the "simplicial size" of ortospheres. In the case of the poset $LP(KG_{m,n})$ we describe the length of the shortest orthocircle, which leads to the following lower bound.

Theorem 50. (Osztényi [28]) *For positive integers m, n, q and $0 \leq r < n$ let l be an integer such that $1 \leq l \leq m - 2n$ and $0 \leq r < ln/(m - 2n)$, then*

$$\chi_{qn-r}(KG_{m,n}) > qm - 2r - l.$$

This gives the multichromatic number $\chi_s(KG_{m,n})$ for the indices $qn - \lfloor \frac{n}{m-2n} \rfloor \leq s \leq qn$. If $m < 3n$ then Theorem 50 proves the Stahl conjecture for previously unknown cases.

Theorem 50 indicates that studying the "simplicial size" of ortospheres in $LP(KG_{m,n})$ we can find further multichromatic numbers $\chi_s(KG_{m,n})$ by more and more sharpening the lower bound for $\chi_s(KG_{m,n})$. To determine the "simplicial size" of ortosphere presents difficulty in the higher dimensional cases.

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