Extremal problems on planar point sets

Abstract of Ph.D. thesis

Viola Mészáros

Supervisors:

Péter Hajnal, University of Szeged
Pavel Valtr, Charles University in Prague

Doctoral School in Mathematics and Computer Science
University of Szeged, Bolyai Institute

Szeged, 2011

Introduction

One of the basic problems in geometric graph theory is to decide if a given graph can be drawn on a given planar point set using pairwise noncrossing straight line edges. In a more demanding version, the points and the vertices of the graph are colored and each vertex has to be placed in a point of the same color (see the survey [8] for further references). Interesting and nontrivial questions arise already if we want to embed a 2-colored path on a 2-colored point set. The authors of several papers have focused on embeddings of so-called alternating paths, which are paths with no monochromatic edge.

Consider an arbitrary $2n$-element equicolored (n points red and n points blue) point set in the plane. We would like to determine or estimate the number of points on the longest noncrossing path such that edges join points of different color and are straight line segments.

In the general case there is still not so much known. If the color classes are separated by a line, then there is a noncrossing, alternating Hamiltonian path on the point set [1]. The same result holds if one of the color classes is exactly the set of vertices of the convex hull [1]. If the color classes are not separated by a line, then there are point sets with no noncrossing, alternating Hamiltonian path for $n \geq 8$, even if the points are in convex position. By the existence of halving lines the result in [1] yields at least n points on the longest noncrossing, alternating path for any equicolored point set of $2n$ points.

Erdős [6] asked what happens if we restrict the points to be in convex position.

$$\ell(\mathcal{P}) = \max_{U \textit{ is a noncrossing alternating path}} \ell(U),$$

where $\ell(U)$ is the number of points on U.

$$\ell(n) = \min_{\mathcal{P} \textit{ is equicolored}} \ell(\mathcal{P}),$$

where $\mathcal{P}$ is any colored planar $2n$-element convex point set.

Without loss of generality we may assume that the points are on a circle. Erdős conjectured that the following configuration was asymptotically extremal. Let n be divisible by four. Divide the circle into four intervals that consist of $\frac{n}{2}$ red, $\frac{n}{4}$ blue, $\frac{n}{2}$ red and $\frac{3n}{4}$ blue points, respectively. In this configuration there are $\frac{3n}{2} + 2$ points on the longest noncrossing, alternating path.

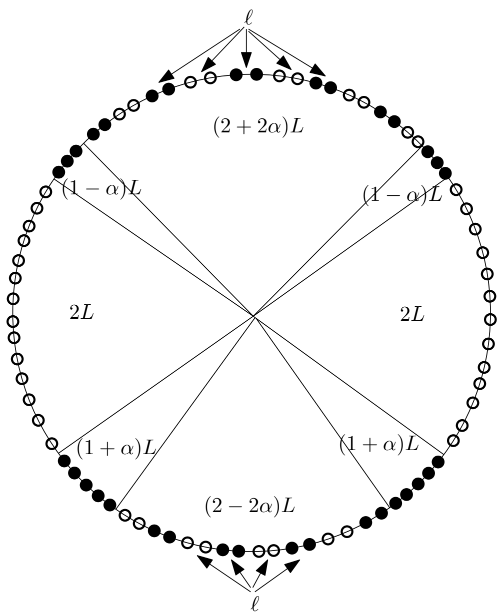

Figure 1: The coloring $\mathcal{P}_{\alpha,\ell}$ where ℓ denotes the common length of the short monochromatic arcs in the two middle arcs (here $\ell = 2$)

Kynčl, Pach and Tóth [10] disproved the above conjecture with a single construction in 2008 and gave the $\frac{4}{3}n + O(\sqrt{n})$ upper and the $n + \Omega\left(\sqrt{n/\log n}\right)$ lower bound. The upper bound is conjectured to be asymptotically tight.

Results

In my Ph.D. thesis I present a class of configurations exihibiting the upper bound $\frac{4}{3}n + O(\sqrt{n})$ [7] in a convex point set for the number of points on noncrossing, alternating paths. This result is joint work with my advisor Péter Hajnal.

This class of configurations can be described as follows. Let $\mathcal{P}_{\alpha,\ell}$ be a coloring of the $2n = 12L$ equicolored point set where $\alpha \in [-1, 1]$ and ℓ is the length of certain consecutive monochomatic arcs, see Figure 1. We assume that αL is an integer. On Figure 1 the number of points of monochromatic and mixed colored arcs in the coloring $\mathcal{P}_{\alpha,\ell}$ is shown in the central angle of each arc.

We proved the following result:

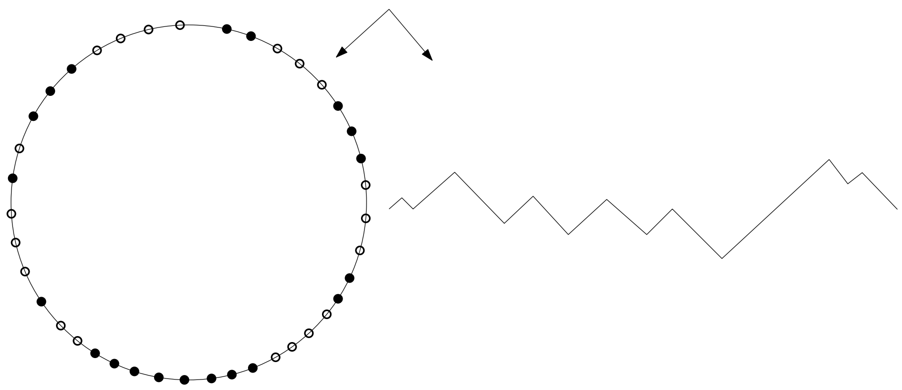

Figure 2: How to code a colored point set as a Dyck path

Theorem 1. *[7] If* $\ell = \Theta(\sqrt{n})$*, then* $\ell(\mathcal{P}_{\alpha,\ell}) = \frac{4n}{3} + O(\sqrt{n})$*.*

This class was also found independently by Jan Kynčl [9] using computer search.

The essence of the proof of the lower bound in [10] is a clever way to define an arc so that it is unbalanced (contains significantly more points from one of the color classes) while it is assumed that the alternation between the colors is small along the circle. We did the same using a completely different idea and we obtained a better result [7].

The basic idea of our improvement is a simple coding of the colored point set. We code our point set as a Dyck path, that is, we introduce for each red point a unit *up* line segment and for each blue point we introduce a unit *down* line segment, see an example for the coloring and the coding on Figure 2. The height of the walk reflects how the colors are alternating. Since we code an equicolored point set the walk ends at the level of starting. We cut the closed walk at any point that belongs to the lowest level. This way we obtain a Dyck path coding our colored point set.

Actually, our code contains all the combinatorial information we need to consider the problem. Our Dyck path has $2n$ steps. Each step starts on a level and ends at a neighboring level.

After using some tricky arguments we can locate an unbalanced arc and we obtain the following theorem:

Theorem 2. *[7]* $\ell(n) \geq n + \Omega(\sqrt{n})$*.*

The proof techniques introduced the notion of *separated matchings*, that is, matchings where no two edges cross geometrically and all edges can be crossed by a line.

If someone considers the observed examples in the literature, then the number of points on the longest noncrossing alternating path is between n and $2n$, while the number of alternations among the colors is $o(n)$. Note, if the number of alternations is linear in n, then the longest noncrossing, alternating path gets "long". If the number of alternations is $o(n)$, then the existence of a long noncrossing, alternating path implies the existence of a large separated matching. The existence of a large separated matching always implies the existence of a long noncrossing, alternating path. Hence, we should concentrate on separated matchings.

I gave several new constructions that allow at most $\frac{4}{3}n + O(\sqrt{n})$ points in any separated matching [12]. Among them there was a class of configurations that significantly differs from all known previous constructions. I also presented a type of coloring such that among these colorings in the optimal one any separated matching contains at most $\frac{4}{3}n + O(\sqrt{n})$ points.

I will describe two main constructions and then I give another one by generalizing one of them.

The *size* of a separated matching is the number of points in it. An (as, bs) block consists of a red arc of as points and a blue arc of bs points. An $s(b, a)$ block consists of a red arc of length b followed by a blue arc of length a and this $a + b$ colored points are repeated s many times.

The first construction is $C_1(s, t)$: Take t consecutive $(s, 2s)$ blocks on the circle followed by t many $s(2, 1)$ blocks.

The second construction is ${C_1}^+(a, b, s, t)$: Take t consecutive (as, bs) blocks on the circle followed by t many $s(b, a)$ blocks. Note that ${C_1}^+(1, 2, s, t) = C_1(s, t)$.

The third construction is a class of coloring $\mathcal{C}_2(s, t)$: Take t many $(s, 2s)$ blocks and t many $(s, s(1, 1))$ blocks in an arbitrary order along C.

I proved the following results:

Theorem 3. *[12] In $C_1(s, t)$ the size of every separated matching is at most $\frac{4}{3}n + O(s + t)$.*

Theorem 4. *[12] In ${C_1}^+(a, b, s, t)$ the ratio of the size of the largest separated matching to the total number of points is*

$$\max\left\{\frac{2\min\{a, b\}}{a + b}, \frac{\max\{a, b\}}{a + b}\right\} + O\left(\frac{(a + b)(s + t)}{n}\right).$$

It follows that the order of magnitude of the size of the largest separated matching is at least $\frac{4}{3}n$. Equality occurs when $\max\{a, b\} = 2\min\{a, b\}$. So $C_1(s, t)$ is optimal among ${C_1}^+(a, b, s, t)$.

Theorem 5. *[12] Let C_2 be any coloring from $\mathcal{C}_2(s,t)$. Then the size of every separated matching in C_2 is at most $\frac{4}{3}n + O(s+t)$.*

Theorem 6. *[12] Let C_3 be that coloring from $\mathcal{C}_2(1000,t)$ where the reddish and bluish blocks alternate. Then size of the largest separated matching in C_3 is at least $1.34n$.*

If a and b are constants in $C_1^+(a,b,s,t)$, we can think about our theorems in the following way. Since $s \cdot t = O(n)$, we can choose s and t so that $s, t = O(\sqrt{n})$ and the order of magnitude of $O(s+t)$ becomes negligible. If a and b are not constants, then for a suitable choice we can achieve that the number of alternations is $o(n)$ and at the same time the remainder term is $o(n)$, too which leads to new constructions with short noncrosssing, alternating paths.

The sixth theorem is an exception, there we choose a setting where s is a large constant and t is $\epsilon \cdot n$. So $O(s+t)$ is very small but not negligible. The reason for choosing such a setting is that in C_3 the discrepancy of the coloring is constant (2000). At the same time the size of the optimal matching is very close to the conjectured value.

There is a conjecture related to separated matchings which would be interesting to prove.

Conjecture. [7] Every equicolored convex point set of $2n$ points admits a separated matching of size $\frac{4}{3}n + O(\sqrt{n})$.

I was also investigating point sets with small *discrepancy* coloring, that is, point sets where the difference in cardinality of color classes is bounded from above on any interval of consecutive points. Separated matchings are simpler than alternating paths. Each alternating path consists of a separated macthing part and side edges which are connected to form a path. Small discrepancy itself means many alternations among the two colors, and that alone guarantees a long noncrossing, alternating path. It is a further advantage of separated matchings that we can consider cases with small discrepancy. We believe it might lead to a better understanding of the problem.

Theorem 7. *[12] For any coloring with dicrepancy $d \leq 3$ there is a separated matching of size at least $\frac{4n}{3}$.*

Unfortunately, already the case of dicrepancy three is rather technical to prove by the used methods. The proof is based on paring of the arcs of different color. It could be feasible to find a better pairing algorithm and broaden the result.

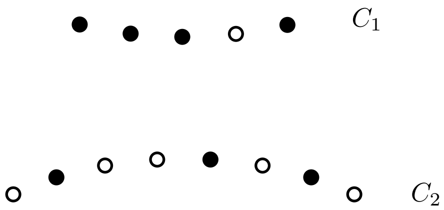

Figure 3: An equicolored double-chain (C_1, C_2)

With my advisor's Pavel Valtr's research group in Prague we considered a specific position of points. Our point set was on a double-chain which we defined in the following way. A *convex* or a *concave chain* is a finite set of points in the plane lying on the graph of a strictly convex or a strictly concave function, respectively. A *double-chain* consists of a convex chain and a concave chain such that any line determined by any of the chains does not intersect the other chain, see Figure 3.

With these settings in 2008 we proved (Cibulka, Kynčl, Mészáros, Stolař, Valtr) the following result:

Theorem 8. *[4] If both chains of the double-chain contain at least one fifth of all the points, then there exists a noncrossing, alternating Hamiltonian path. On the other hand, the above property does not hold if one of the chains contains at most* $\approx 1/29$ *of all the points.*

In the area of long noncrossing, alternating paths on a 2-colored point set there remain more open questions still. The gap is remarkable between the best lower and upper bound gained so far in the convex case. The general case would be also an iteresting line of research.

Finally, I would like to describe the results in the last part of my thesis. With Pavel Valtr's research group we settled a conjecture of Peter Winkler [5]. The problem is the following. Bob cuts a pizza into slices of not necessarily equal size and shares it with Alice by alternately taking turns. One slice is taken in each turn. The first turn is Alice's. She may choose any of the slices. In all other turns only those slices can be chosen that have a neighbor slice already taken. How much of the pizza can Alice gain? Peter Winkler conjectured that Alice can obtain 4/9 of the pizza for any cutting.

The pizza after Bob's cutting may be represented by a circular sequence $P = p_0 p_1 \dots p_{n-1}$ and by *the sizes* $|p_i| \geq 0$ (for $i = 0, 1, \dots, n-1$). For $1 < j \leq n$, if one of the players chooses a slice p_i in the $(j-1)$-st turn and

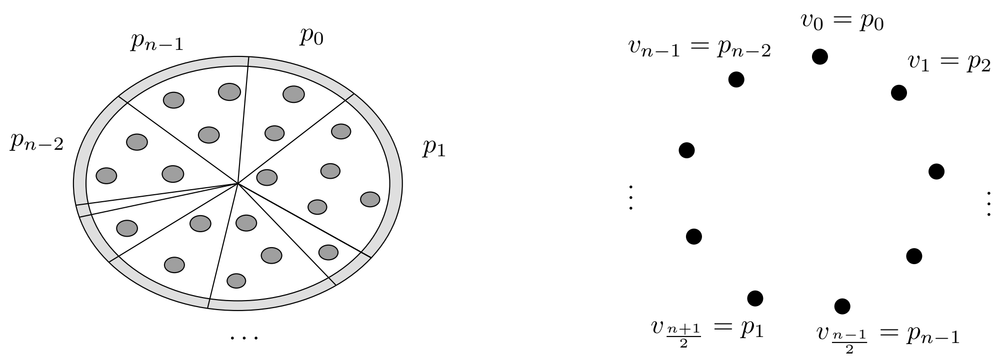

Figure 4: A cutting of a pizza and the corresponding characteristic cycle.

the other player chooses p_{i-1} or p_{i+1} in the j-th turn, then the j-th turn is called a *shift*, otherwise it is called a *jump*.

If the number of slices is even, we can easily argue that Alice gains at least the half of the pizza. If the number of slices is odd, instead of the circular sequence $P = p_0p_1 \dots p_{n-1}$ we will be working with the related circular sequence $V = v_0v_1 \dots v_{n-1} = p_0p_2 \dots p_{n-1}p_1p_3 \dots p_{n-2}$ that we call *the characteristic cycle*, see Figure 4.

Here is our main result:

Theorem 9. *[5] For any P, Alice has a two-jump strategy with gain $4|P|/9$.*

More generally, we determine Alice's guaranteed gain for any given number of slices.

Theorem 10. *[5] For $n \geq 1$, let $g(n)$ be the maximum $g \in [0, 1]$ such that for any cutting of the pizza into n slices, Alice has a strategy with gain $g|P|$. Then*

$$g(n) = \begin{cases} 1 & \text{if } n = 1, \\ 4/9 & \text{if } n \in \{15, 17, 19, \dots\}, \\ 1/2 & \text{otherwise.} \end{cases}$$

Moreover, Alice has a zero-jump strategy with gain $g(n)|P|$ when n is even or $n \leq 7$, she has a one-jump strategy with gain $g(n)|P|$ for $n \in \{9, 11, 13\}$, and she has a two-jump strategy with gain $g(n)|P|$ for $n \in \{15, 17, 19, \dots\}$.

If we make a restriction on the number of Alice's jumps we get the following results.

Theorem 11. *[5]* (a) *Alice has a zero-jump strategy with gain $|P|/3$ and the constant $1/3$ is the best possible.*

(b) *Alice has a one-jump strategy with gain* $7|P|/16$ *and the constant* $7/16$ *is the best possible.*

Due to Theorem 10, the following theorem describes all minimal cuttings for which Bob has a strategy with gain $5|P|/9$.

Theorem 12. *[5] For any* $\omega \in [0,1]$*, Bob has a one-jump strategy with gain* $5|P|/9$ *if he cuts the pizza into* 15 *slices as follows:* $P_\omega = 0010100(1+\omega)0(2-\omega)00202$*. These cuttings describe, up to scaling, rotating and flipping the pizza upside-down, all the pizza cuttings into* 15 *slices for which Bob has a strategy with gain* $5|P|/9$*.*

For $\omega = 0$ or $\omega = 1$, the cutting in Theorem 12 has slices of three different sizes $0, 1, 2$. If all the slices have the same size, then Alice always gets at least half of the pizza. But two different slice sizes are already enough to obtain a cutting with which Bob gets 5/9 of the pizza.

Theorem 13. *[5] Up to scaling, rotating and flipping the pizza upside-down, there is a unique pizza cutting into* 21 *slices of at most two different sizes for which Bob has a strategy with gain* $5|P|/9$*. The cutting is* 001010010101001010101*.*

We describe a linear-time algorithm for finding Alice's two-jump strategy with gain $g(n)|P|$ guaranteed by Theorem 10.

Theorem 14. *[5] There is an algorithm that, given a cutting of the pizza with* n *slices, performs a precomputation in time* $O(n)$*. Then, during the game, the algorithm decides each of Alice's turns in time* $O(1)$ *in such a way that Alice makes at most two jumps and her gain is at least* $g(n)|P|$*.*

Using the ideas of the proofs above it is also straightforward to present an efficient algorithm for finding optimal strategies for each of the two players. The algorithm design uses dynamic programing and its running time is quadratic.

Claim 15. *[5] There is an algorithm that, given a cutting of the pizza with* n *slices, computes an optimal strategy for each of the two players in time* $O(n^2)$*. The algorithm stores an optimal turn of the player on turn for all the* $n^2 - n + 2$ *possible positions of the game.*

Bibliography

[1] M. Abellanas, J. García, G. Hernandez, M. Noy, P. Ramos, Bipartite embeddings of trees in the plane, Discrete Appl. Math. 93 (1999), 141–148.

[2] M. Abellanas, A. Garcia, F. Hurtado, and J. Tejel: Caminos alternantes, in: X Encuentros de Geometria Computacional (in Spanish), Sevilla, 2003, 7–12.

[3] J. Cibulka, J. Kynčl, V. Mészáros, R. Stolař, P. Valtr, Graph sharing games: complexity and connectivity, submitted. Extended abstract in: TAMC 2010, Lecture Notes in Computer Science 6108, 340-349, Springer, Berlin, 2010.

[4] J. Cibulka, J. Kynčl, V. Mészáros, R. Stolař, P. Valtr, Hamiltonian alternating paths on bicolored double-chains, in: I. G. Tollis, M. Patrignani (Eds.), Graph Drawing 2008, Lecture Notes in Computer Science 5417, Springer, New York, 2009, pp. 181–192.

[5] J. Cibulka, J. Kynčl, V. Mészáros, R. Stolař, P. Valtr, The Solution of Peter Winkler's Pizza Problem, Fete of Combinatorics, Bolyai Society Mathematical Studies, Vol. 20, 63-93, Springer, 2010. Extended abstract was published in Combinatorial Algorithms (proceedings of IWOCA 2009), Lecture Notes in Computer Science 5874, 356-367, Springer, Berlin, 2009.

[6] Paul Erdős, Personal communication to János Pach (see [10])

[7] P. Hajnal, V. Mészáros, A note on noncrossing path in colored convex sets, accepted to Discrete Mathematics and Theoretical Computer Science.

[8] A. Kaneko and M. Kano: Discrete geometry on red and blue poins in the plane — a survey, in: Discrete and Computational Geometry (B. Aronov et al., eds.), Springer-Verlag, Berlin, 2004, 551–570.

[9] Jan Kynčl, Personal communication

[10] J. Kynčl, J. Pach and G. Tóth , Long alternating paths in bicolored point sets in: Graph Drawing (J. Pach, ed.), Lecture Notes in Computer Science 3383, Springer-Verlag, Berlin, 2004, 340-348. Also in: Discrete Mathematics 308 (2008), 4315–4322.

[11] C. Merino, G. Salazar and J. Urrutia, On the length of the longest alternating path for multicoloured point sets in convex position. Discrete Mathematics Vol. 360, no. 15, pp. 1791–1797, 2006.

[12] V. Mészáros, Separated matchings on colored convex sets, submitted.