DOCTORAL THESIS

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1. Orders of permutation groups

Bounding the order of a primitive permutation group in terms of its degree was a problem of 19-th century group theory. Apart from some early results of Jordan probably the first successful estimate for the orders of primitive groups not containing the alternating group is due to Bochert [7] (see also [17] or [50]): if G is primitive and $(S_n : G) > 2$, then $(S_n : G) \geq [\frac{1}{2}(n+1)]!$. This bound is useful since it is the sharpest available general estimate for very small degrees. But it is far from best possible. Based on Wielandt's method [51] of bounding the orders of Sylow subgroups Praeger and Saxl [41] obtained an exponential estimate, 4^n, where n is the degree of the permutation group. Their proof is elaborate. Using entirely different combinatorial arguments, Babai [2] obtained an $e^{4\sqrt{n}\ln^2 n}$ estimate for uniprimitive (primitive but not doubly transitive) groups. For the orders of doubly transitive groups not containing the alternating group, Pyber obtained an $n^{32\log^2 n}$ bound for $n > 400$ in [43] by an elementary argument (using some ideas of [3]). Apart from $O(\log n)$ factors in the exponents, the former two estimates are asymptotically sharp. To do better, one has to use the Aschbacher-O'Nan-Scott theorem and the classification of finite simple groups. An $n^{c \ln\ln n}$ type bound with "known" exceptions has been found by Cameron [14], while an $n^{9\log_2 n}$ estimate follows from Liebeck [29]. In our dissertation we use the classification of finite simple groups to set the sharpest upper bounds possible for the orders of primitive permutation groups via a reasonably short argument. First the following is proved.

Theorem 1.1. *Let G be a primitive permutation group of degree n. Then one of the following holds.*
(i) G is a subgroup of $S_m \wr S_r$ containing $(A_m)^r$, where the action of S_m is on k-element subsets of $\{1, ..., m\}$ and the wreath product has the product action of degree $n = \binom{m}{k}^r$;
(ii) $G = M_{11}$, M_{12}, M_{23} or M_{24} with their 4-transitive action;
(iii) $|G| \leq n \cdot \prod_{i=0}^{[\log_2 n]-1}(n - 2^i) < n^{1+[\log_2 n]}$.

This is a sharp version of the above-mentioned result of Liebeck. The theorem practically states that if G is a primitive group, which is not uniprimitive of case (i), and is not 4-transitive, then the estimate in (iii) holds. The bound in (iii) is best possible. There are infinitely many 3-transitive groups, in particular the affine groups, $AGL(t,2)$ acting on 2^t points and the symmetric group, S_5 acting on 6 points for which the estimate is exact. In fact, these are the only groups among groups not of case (i) and (ii) for which equality holds. But there is one more infinite sequence of groups displaying the sharpness of the bound. The projective groups, $PSL(t,2)$ acting on the $t > 2$ dimensional projective space have order $\frac{1}{2} \cdot (n+1) \cdot \prod_{i=0}^{[\log_2 n]-1} (n+1-2^i) < n \cdot \prod_{i=0}^{[\log_2 n]-1} (n-2^i)$, where $n = 2^t - 1$.

An easy direct consequence is

Corollary 1.1. *Let G be a primitive subgroup of S_n.*

(i) If G is not 3-transitive, then $|G| < n^{\sqrt{n}}$;

(ii) If G does not contain A_n, then $|G| < 50 \cdot n^{\sqrt{n}}$.

This is a sharp version of a result of Cameron [14]. The estimate in (i) is asymptotically sharp for uniprimitive groups of case (i) of Theorem 1.1 and is sharp for the automorphism group of the Fano-plane. The estimate in (ii) is sharp for the biggest Mathieu group. For an application of Corollary 1.1 to semigroup theory see [1]. Theorem 1.1 also leads to a sharp exponential bound.

Corollary 1.2. *If G is a primitive subgroup of S_n not containing A_n, then $|G| < 3^n$. Moreover, if $n > 24$, then $|G| < 2^n$.*

This improves the Praeger-Saxl [41] theorem. The proof displays M_{12} as the "largest" primitive group. M_{24} has order greater than 2^{24}, which explains the requirement $n > 24$ in the latter statement. But let us put this in a slightly different form with the use of the prime number theorem.

Corollary 1.3. *If G is a primitive subgroup of S_n not containing A_n, then $|G|$ is at most the product of all primes not greater than n, provided that $n > 24$.*

In [27] Kleidman and Wales published a list of primitive permutation groups of order at least 2^{n-4}. However their list is rather lengthy, and it is not easy to use. Using our results above we will relax the bound to 2^{n-1} to give a shorter list of "large" primitive groups. These exceptional groups are referred to in [32]. (Note that the Kleidman-Wales list can also be deduced by a similar argument.)

Corollary 1.4. *Let G be a primitive permutation group of degree n not containing A_n. If $|G| > 2^{n-1}$, then G has degree at most 24, and is permutation isomorphic to one of the following 24 groups with their natural permutation representation if not indicated otherwise.*

(i) $AGL(t,q)$ *with* $(t,q) = (1,5)$, $(3,2)$, $(2,3)$, $(4,2)$*;* $A\Gamma L(1,8)$ *and* $2^4 : A_7$*;*

(ii) $PSL(t,q)$ *with* $(t,q) = (2,5)$, $(3,2)$, $(2,7)$, $(2,8)$, $(3,3)$, $(4,2)$*;* $PGL(t,q)$ *with* $(t,q) = (2,5)$, $(2,7)$, $(2,9)$*;* $P\Gamma L(2,8)$ *and* $P\Gamma L(2,9)$*;*

(iii) M_i *with* $i = 10$, 11, 12, 23, 24*;*

(iv) S_6 *with its primitive action on* 10 *points, and* M_{11} *with its action on* 12 *point.*

From the above list, using an inductive argument, one can deduce the theorem of Liebeck and Pyber [30] stating that a permutation group of degree n has at most 2^{n-1} conjugacy classes.

Another possible application of the previous result was suggested in [44] by Pyber. Improving restrictions on the composition factors of permutation groups one can bound their order. For example, Dixon [16] proved that a solvable permutation group of degree n has order at most $24^{(n-1)/3}$, and Babai, Cameron, Pálfy [4] showed that a subgroup of S_n that has no composition factors isomorphic to an alternating group of degree greater than $d (d \geq 6)$ has order at most d^{n-1}. Applying the former results Dixon's theorem may be generalized and Babai-Cameron-Pálfy's estimate may be sharpened as follows.

Corollary 1.5. *Let G be a permutation group of degree n, and let d be an integer not less than 4. If G has no composition factor isomorphic to an alternating group of degree greater than d, then $|G| \leq d!^{(n-1)/(d-1)}$.*

This bound is best possible. If n is a power of d, then the iterated wreath product of n/d copies of S_d has order precisely $d!^{(n-1)/(d-1)}$. The proof shows that the Mathieu group, M_{12} is again of special importance.

For an application of this corollary see chapter 3 of the book by Lubotzky and Segal [32], and for an alternative approach to dealing with nonabelian alternating composition factors see the paper [25] by Holt and Walton.

2. Counting conjugacy classes

In many cases it is more natural to count complex irreducible characters than conjugacy classes for a finite group. However, sometimes it is indeed more natural to work with conjugacy classes. Here is an example.

Nagao [38] proved that if G is a finite group and N is a normal subgroup, then the number of irreducible characters of G is at most the number of irreducible characters of N multiplied by the number of irreducible characters of the factor group, G/N. Later, Gallagher [20] proved this fact using conjugacy classes. We believe that this proof is more natural. If we denote the number of conjugacy classes of a finite group G by $k(G)$, then Nagao's result translates to

Lemma 2.1 (Nagao, [38]). *If G is a finite group and N a normal subgroup in G, then $k(G) \leq k(N) \cdot k(G/N)$.*

This lemma is very important. It was first used in proving that for p-solvable groups, Brauer's $k(B)$-problem is equivalent to the $k(GV)$-problem, and most recently it was used many times in proving the $k(GV)$-problem itself.

Here, we give another application of Lemma 2.1 of a (seemingly) different flavor.

Theorem 2.1 (Kovács, Robinson, [28]). *If G is a permutation group of degree n, then $k(G) \leq 5^{n-1}$.*

The proof is inductive. First we prove the claim for primitive, then for imprimitive and finally for intransitive groups. The induction starts by giving a universal upper bound for the numbers of conjugacy classes for primitive groups, after which we apply Lemma 2.1 in each step of the induction. The initial case is the most difficult one. In the Kovács-Robinson proof the Praeger-Saxl [41] bound on the orders of primitive permutation groups was used. This makes Theorem 2.1 independent of the Classification Theorem of Finite Simple Groups (CTFSG).

However, if one wants to improve on the bound in Theorem 2.1, then CTFSG is necessary. By a result of the previous chapter on the orders of primitive groups, one can give a short proof for

Theorem 2.2 (Liebeck, Pyber, [30]). *If G is a permutation group of degree n, then $k(G) \leq 2^{n-1}$.*

Originally, Theorem 2.2 was proved by estimating the numbers of conjugacy classes of simple groups via existing recurrence relations for these numbers. (Indeed, Kovács and Robinson had verified a relevant reduction to almost simple groups.) Recently, the Liebeck-Pyber bound for simple groups (of Lie-type) was (and is being) improved by Fulman and Guralnick in [19] and in a paper in preparation. In our dissertation we make no use of these improvements in proving

Theorem 2.3. *Any permutation group of degree $n > 2$ has at most $3^{(n-1)/2}$ conjugacy classes.*

This theorem was used in [21].

3. Covering the symmetric groups with proper subgroups

Let G be a group that is a set-theoretic union of finitely many proper subgroups. Cohn [15] defined the function $\sigma(G)$ to be the least integer m such that G is the union of m of its proper subgroups. (A result of Neumann [39] states that if G is the union of m proper subgroups where m is finite and small as possible, then the intersection of these subgroups is a subgroup of finite index in G. Hence in investigating σ we may assume that G is finite.) It is an easy exercise that $\sigma(G)$ can never be 2; it is at least 3. Groups that are the union of three proper subgroups, as $C_2 \times C_2$ is for example, are investigated in the papers [46], [22], and [12]. Moreover, $\sigma(G)$ can be 4, 5, and 6 too, as the examples, $C_3 \times C_3$, A_4, and $C_5 \times C_5$ show. However, Tomkinson [49] proved that there is no group G with $\sigma(G) = 7$. Cohn [15] showed that for any prime power p^a there exists a solvable group G with $\sigma(G) = p^a + 1$. In fact, Tomkinson [49] established that $\sigma(G) - 1$ is always a prime power for solvable groups G. He also pointed out that it would be of interest to investigate σ for families of simple groups. Indeed, the situation for nonsolvable groups seems to be totally different. Bryce, Fedri, Serena [13] investigated certain nonsolvable 2-by-2 matrix groups over finite fields, $((P)G(S)L(2,q))$ and obtained the formula $\frac{1}{2}q(q+1)$ for even prime powers $q \geq 4$, and the formula $\frac{1}{2}q(q+1)+1$ for odd prime powers $q \geq 5$. Moreover, Lucido [33] found that $\sigma(Sz(q)) = \frac{1}{2}q^2(q^2+1)$ where $q = 2^{2m+1}$. There are partial results due to Bryce and Serena for determining $\sigma((P)G(S)L(n,q))$.

The following is established.

Theorem 3.1. *Let $n > 3$, and let S_n and A_n be the symmetric and the alternating group respectively on n letters.*

(1) We have $\sigma(S_n) = 2^{n-1}$ *if* n *is odd unless* $n = 9$*, and* $\sigma(S_n) \leq 2^{n-2}$ *if* n *is even.*

(2) If $n \neq 7, 9$*, then* $\sigma(A_n) \geq 2^{n-2}$ *with equality if and only if* n *is even but not divisible by* 4*.*

In our dissertation we prove more than what is stated in Theorem 3.1. We will obtain exact or asymptotic formulas in all (infinite) cases (possibly) except for $\sigma(A_p)$ where p is a prime of the form $(q^k-1)/(q-1)$ where q is a prime power and k is a positive integer.

For the groups S_9, S_{12}, A_7, and A_9 we only prove $172 \leq \sigma(S_9) \leq 256$, $\sigma(S_{12}) \leq 761$, $\sigma(A_7) \leq 31$, and $\sigma(A_9) \geq 80$. Notice that the numbers 761 and 31 are primes not of the form $q + 1$ where q is a prime power. We prove that $\sigma(G)$ can indeed be such a prime.

Proposition 3.1. *For the smallest Mathieu group we have* $\sigma(M_{11}) = 23$*.*

This result was also proved (independently) by Holmes in [23]. In her paper many interesting results are found about sporadic simple groups. It is proved that $\sigma(M_{22}) = 771$, $\sigma(M_{23}) = 41079$, $\sigma(O'N) = 36450855$, $\sigma(Ly) = 112845655268156$, $5165 \leq \sigma(J_1) \leq 5415$, and that $24541 \leq \sigma(McL) \leq 24553$.

At this point we note that Tomkinson [49] conjectured that $\sigma(G)$ can never be 11 nor 13.

The commuting graph Γ of a group G is as follows. Let the vertices of Γ be the elements of G and two vertices g, h of Γ are joined by an edge if and only if g and h commute as elements of G. (The commuting graph is used to measure how abelian the group is. See [18], and [45].) Several people have studied $\alpha(G)$, the maximal cardinality of an empty subgraph of Γ and $\beta(G)$, the minimal cardinality of a covering of the vertices of Γ by complete subgraphs. (See the papers [18], [37], and [42].) Brown investigated the relationship between the numbers $\alpha_n = \alpha(S_n)$ and $\beta_n = \beta(S_n)$. In [10] it is shown that these numbers are surprisingly close to each other, though for $n \geq 15$, they are never equal [11].

As an application of Theorem 3.1, we prove that if we add 'more' edges to the commuting graph of the symmetric group, then the corresponding numbers will be equal in infinitely many cases. Let G be a group. Define a graph Γ' on the elements of G with the property that two group elements are joined by an edge if and only if they generate a proper subgroup of G. Similarly as for the commuting graph, we may define $\alpha'(G)$ and $\beta'(G)$ for our new graph, Γ'. Put $\alpha'_n = \alpha'(S_n)$ and $\beta'_n = \beta'(S_n)$. The theorem can now be stated.

Theorem 3.2. *There is a subset S of density 1 in the set of all primes so that $\alpha'_n = \beta'_n$ holds for all $n \in S$.*

The equality $\alpha'_n = \beta'_n$ is valid for very small values of n also. Does it hold for all n?

We note that the problem of covering groups by subgroups has found interest for many years. The first reference the author is aware of is the 1926 work of Scorza [46]. Probably Neumann [39], [40] was the first to study the number of (abelian) subgroups needed to cover a (not necessarily finite) group G in relation to the index of the center of G. For a survey of this area see [47]. On the other hand, for an extensive account of work in (packing and) covering groups with (isomorphic) subgroups (or of subgroups of a specified order) the reader is referred to [26].

4. Sets of elements that pairwise generate a linear group

Let G be a finite group that can be generated by two elements. We define $\mu(G)$ to be the largest positive integer m so that there exists a subset X in G of order m with the property that any distinct pair of elements of X generates G. Let n be a positive integer, q a prime power, and V the n-dimensional vector space over the field of q elements. Let $[x]$ denote the integer part of the real number x. We have

Theorem 4.1. *Let G be any of the groups $(P)GL(n,q)$, $(P)SL(n,q)$. Let b be the smallest prime factor of n, and let $N(b)$ be the number of proper subspaces of V of dimensions not divisible by b. If $n \geq 12$, then*

$$\mu(G) = \frac{1}{b}\prod_{\substack{i=1 \\ b \nmid i}}^{n-1}(q^n - q^i) + [N(b)/2].$$

Let S_n be the symmetric group on n letters. In the previous chapter it was stated that the set of prime numbers n for which $\mu(S_n) = \sigma(S_n) = 2^{n-1}$ has density 1 in the set of all primes. In a beautiful paper, this result was considerably extended by Blackburn [6] who showed that if n is a sufficiently large odd integer, then $\mu(S_n) = \sigma(S_n) = 2^{n-1}$, and that if n is a sufficiently large integer congruent to 2 modulo 4, then $\mu(A_n) = \sigma(A_n) = 2^{n-2}$ for the alternating group A_n. In the same paper Blackburn asked what the relationship between the numbers $\sigma(G)$ and

$\mu(G)$ is when G is a finite simple group. For example, is it true that $\sigma(G)/\mu(G) \to 1$ as $|G| \to \infty$? An affirmative answer to this question in the special case when G is a projective special linear group is given in Section 6 of [9]. In many cases we can say more.

Theorem 4.2. *Let G be any of the groups $(P)GL(n,q)$, $(P)SL(n,q)$. Let b be the smallest prime factor of n, let $\left[{n \atop k}\right]_q$ be the number of k-dimensional subspaces of the n-dimensional vector space V, and let $N(b)$ be the number of proper subspaces of V of dimensions not divisible by b. Suppose that $n \geq 12$. Then if $n \not\equiv 2 \pmod 4$, or if $n \equiv 2 \pmod 4$, q odd and $G = (P)SL(n,q)$, then*

$$\sigma(G) = \mu(G) = \frac{1}{b} \prod_{\substack{i=1 \\ b \nmid i}}^{n-1} (q^n - q^i) + [N(b)/2].$$

Otherwise $\sigma(G) \neq \mu(G)$ and

$$\sigma(G) = \frac{1}{2} \prod_{\substack{i=1 \\ 2 \nmid i}}^{n-1} (q^n - q^i) + \sum_{\substack{k=1 \\ 2 \nmid k}}^{(n/2)-1} \begin{bmatrix} n \\ k \end{bmatrix}_q + \frac{q^{n/2}}{q^{n/2}+1} \begin{bmatrix} n \\ n/2 \end{bmatrix}_q + \epsilon$$

where $\epsilon = 0$ if q is even and $\epsilon = 1$ if q is odd.

Theorem 4.2 extends earlier results of Bryce, Fedri, Serena [13]. Also, the formulae for $\sigma(G)$ for the groups $(P)GL(3,q)$, $(P)SL(3,q)$ was kindly communicated to one of us by Serena [48].

A couple of remarks need to be made.

A quick corollary to the solution of Dixon's conjecture, stated by Liebeck and Shalev in [31] (see Corollary 1.7), is that there exists a universal constant c so that $\mu(G) \geq c \cdot n$ for any finite simple group G where n denotes the minimal index of a proper subgroup in G.

A group is said to have spread at least k if, for any non-identity $x_1, \ldots, x_k \in G$, there is some $y \in G$ such that $G = \langle x_i, y \rangle$ whenever $1 \leq i \leq k$. The number $s(G)$ denotes the largest integer k so that G has spread at least k. There are many papers on spread, see, for example, Breuer, Guralnick, Kantor [8]. It is easy to see that for any non-cyclic finite group G that can be generated by two elements, the inequality $s(G) < \mu(G)$ holds.

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