ON STABILITY CONDITIONS OF OPERATOR SEMIGROUPS

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1. Introduction

1.1. **ABLV type theorems.** Stability theorems play an important part in the theory of linear operator semigroups. Let A be a densely defined closed operator on a complex Banach space $\mathcal{X}$. The solutions of the well-posed abstract Cauchy problems

$$\text{(ACP)} \quad \begin{cases} \dot{u}(t) &= Au(t) \quad (0 \leq t), \\ u(0) &= x, \end{cases}$$

form an operator semigoup $(T(t))_{t\geq 0}$ on $\mathcal{X}$ (see [11]). The well-known Arendt–Batty–Lyubich–Vũ theorem [32] gives a sufficient spectral condition for the stability of the operator semigroup. Let $\sigma(A)$ and $\sigma_{\mathrm{p}}(A^*)$ stand for the spectrum of A and the point spectrum of its adjoint A^*.

Theorem 1.1. (ABLV) *Let $(T(t))_{t\geq 0}$ be the C_0-semigroup generated by A and suppose that $T(t)x$ is bounded for every $x \in \mathcal{X}$. If $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is countable and $\sigma_{\mathrm{p}}(A^*) \cap \mathrm{i}\mathbb{R} = \emptyset$, then*

$$\lim_{t\to\infty} \|T(t)x\| = 0$$

for every $x \in \mathcal{X}$.

Quite a few generalizations of the theorem are known for bounded and unbounded representations of suitable locally compact abelian semigroups ([1], [2], [5], [3], [19], [20]). First Vũ [41] proved a weighted version of the ABLV Theorem. Later, C.J.K. Batty and S. Yeates [3] gave a detailed study on the spectral theory and stability of non-quasianalytic representations. In this thesis, we shall extend Kérchy's method (which appears in papers [19], [20], applied to discrete abelian semigroups) to topological semigroups; that is, we shall prove stability results for unbounded representations that have a regular norm-function. The derived stability results are strongly related to [3], but the main differences lie in the stability properties and the norm-conditions of the semigroup. In addition, on the real half line, we shall give a characterization of C_0-semigroups whose norm-function is topologically regular. We shall discuss these results in Sections 2 to 5. Different applications of regularity related to generalized Toeplitz operators and similarity problems can be found in [23], [24] and [7], [8].

1.2. **Katznelson–Tzafriri type theorems.** Another type of stability result studied in this thesis is the Katznelson–Tzafriri theorem (see [18]). Let $A(\mathbb{T})$ denote the set of continuous functions on the unit circle $\mathbb{T}$ whose Fourier coefficients are absolutely convergent, and let $A^+(\mathbb{T})$ be the set of functions in $A(\mathbb{T})$ whose Fourier coefficients with negative indeces are vanishing. The algebra $A(\mathbb{T})$ is a Banach algebra with the norm $\|f\| = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|$ (where $f \in A(\mathbb{T})$ and $\{\widehat{f}(n)\}_{n=1}^{\infty}$ are the Fourier coefficients of f). We say that an $f \in A^+(\mathbb{T})$ is of spectral synthesis with respect to a closed set $E \subseteq \mathbb{T}$ if there exists a sequence of $(f_n)_n \subset A(\mathbb{T})$ such that each f_n vanishes in a neighbourhood of E and $\lim_{n\to\infty} \|f_n - f\| = 0$. The Katznelson–Tzafriri theorem is the following.

Theorem 1.2. *Let T be a power-bounded operator on the Banach space $\mathcal{X}$, and let $f \in A^+(\mathbb{T})$, which is of spectral synthesis with respect to $\sigma(T) \cap \mathbb{T}$, the peripheral spectrum of T. Then*

$$\lim_{n\to\infty} \|T^n f(T)\| = 0.$$

We note that for Hilbert space contractions a richer functional calculus can be defined due to von Neumann's inequality. It can be proved [13] that if f is an element of the disk algebra $A(\mathbb{D})$ (i.e. f is analytic on the open unit disk $\mathbb{D}$ and it can be extended continuously to the closed unit disk) then f vanishes on the peripheral spectrum of T if and only if $\lim_{n\to\infty} \|T^n f(T)\| = 0$. On the other hand, contractions even admit a more general H^∞ calculus on the unit disk, the so-called Sz.Nagy–Foias calculus. Let T be a completely nonunitary contraction on a Hilbert space and let f be a bounded holomorphic function on the unit disk. Then the convergence $\lim_{n\to\infty} \|T^n f(T)\| = 0$ holds if $\lim_{r\to 1} f(re^{i\theta}) = 0$ for every $e^{i\theta}$ in the peripheral spectrum of T. However, an example shows that the converse implication is not true (see [6]).

We will prove that the assumption made in the Katznelson–Tzafriri theorem can be weakened in Hilbert spaces, and we will provide a complete characterization of the convergence $\lim_{n\to\infty} \|T^n Q\| = 0$ whenever Q commutes with T. This result shall be presented in Section 6.

Many extensions of the Katznelson–Tzafriri were proved in the discrete case as well as in the continuous one; see [5], [14], [17], [33], [34], [40] and [2], [9]. However, we recall that all former extensions of the Katznelson–Tzafriri theorem are related to bounded functional calculi of T or elements of the Banach algebra generated by T.

2. Amenability on semigroups

Consider a locally compact, Hausdorff abelian group $(G;+)$. Let S be a closed subsemigroup of G with non-empty interior S° such that $S-S=G$ and $S\cap(-S) = \{0\}$. By definition, for any $s_1, s_2 \in S$, $s_1 \preceq s_2$ if $s_2 - s_1 \in S$. In this way we obtain an inductive partial ordering on S. Let μ denote the restriction of the Haar measure $\tilde{\mu}$ on G to S. We shall use the notation $L^\infty(S)$ for the Banach space of essentially bounded, measurable functions with respect to μ on S. The translation of a function $f\colon S \to \mathbb{C}$ by $s \in S$ is the mapping $f_s\colon S \to \mathbb{C}$ defined by $f_s(s') := f(s+s')$ $(s' \in S)$.

We say that a functional m in the dual space $L^\infty(S)^*$ is an invariant mean, if

- $\|m\| = m(\mathbf{1}) = 1$,
- $m(f_s) = m(f)$ for every $s \in S$.

(Here $\mathbf{1}$ denotes the constant 1 function on S).

It can be shown that the set of the invariant means defined on S is non-empty. This set shall be denoted by $\mathcal{M}(S)$. In general, a (not necessarily abelian) semigroup or group is said to be amenable if there exists an invariant mean on it. A sufficient and necessary condition of amenability can be given using (strong) Følner nets.

Definition 2.1. A net $\{K_\lambda\}_{\lambda\in\Lambda}$ of compact subsets of G with nonempty interior is a *strong Følner net* if

(i) $K_{\lambda_1} \subseteq K_{\lambda_2}$ whenever $\lambda_1 \preceq \lambda_2$,
(ii) $G = \bigcup K_\lambda^\circ$,
(iii) $\tilde{\mu}((x+K_\lambda) \triangle K_\lambda)/\tilde{\mu}(K_\lambda) \to 0$ (as $\lambda \to \infty$) uniformly when x runs through compact sets. (Here and in the following $\triangle$ stands for the symmetric difference.)

A net $\{K_\lambda\}_{\lambda\in\Lambda}$ of compact subsets of G with nonempty interior is called a *Følner net* for G only if property (iii) holds.

For example, if $G = \mathbb{R}$ then $K_n := [-n, n]$ is a Følner sequence. By the characterization theorem of amenable groups ([36, Theorem 4.16]), a locally compact group G is amenable if and only if there exists a strong Følner net for G. If G is σ-compact, then we can always find a strong Følner sequence in G.

By means of the Markov–Kakutani fixed point theorem, one can prove in a simple way that locally compact abelian (semi)groups are amenable; thus there exists a Følner net on them. Shifting the elements of a net, the Følner net will lie in the interior of S.

2.1. **Topologically invariant means.** In the thesis, we shall use also a special subset of invariant means. Topologically invariant means were originally introduced by A. Hulanicki for locally compact Hausdorff groups (see [36, page 9]). We shall apply his concept to semigroups. First let $\mathcal{G}(S)$ denote the set of non-negative, measurable functions g on S which satisfy the condition $\int_S g(s)\, ds = 1$. Next, for any $f \in L^\infty(S)$ and $g \in \mathcal{G}(S)$, let us consider a convolution $f * g \in L^\infty(S)$, defined by $(f * g)(y) = \int_S f(s + y)g(s)\, d\mu(s)$.

Definition 2.2. We say that a functional $m \in L^\infty(S)^*$ is a *topologically invariant mean* if

- $\|m\| = m(\mathbf{1}) = 1$,
- $m(f * g) = m(f)$ for every $f \in L^\infty(S)$ and $g \in \mathcal{G}(S)$.

One can verify that the set of the topologically invariant means $\mathcal{M}_{\mathrm{t}}(S)$ is nonempty. A little reasoning shows that every topologically invariant mean m is a (translation) invariant mean. Indeed, for any fixed $f \in L^\infty(S)$ and $y \in S$, let us choose a function $g \in \mathcal{G}(S)$ such that the support of g is included in $y + S$. We have

$$m(f) = m(f * g) = m\left(\int_{y+S} f(\,\cdot + s)g(s)\, d\mu(s)\right) = m(f_y * g_y) = m(f_y)$$

because g_y is in $\mathcal{G}(S)$, so $\mathcal{M}_{\mathrm{t}}(S) \subseteq \mathcal{M}(S)$.

In the following example, we will give a construction which shows that the inclusion $\mathcal{M}_{\mathrm{t}}(S) \subseteq \mathcal{M}(S)$ can be strict; that is, there exists an invariant mean that is not topologically invariant.

Let $r_1, r_2, \dots$ be an enumeration of the non-negative rational numbers. With $\Omega = \bigcup_{n=1}^{\infty}(r_n - 2^{-n}, r_n + 2^{-n})$, let f_0 be the characteristic function of $\Omega \cap \mathbb{R}_+$. Since $f_0 \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, we get that $m(f_0) = 0$ for every $m \in \mathcal{M}_{\mathrm{t}}(\mathbb{R}_+)$. However, the following holds true [28, Proposition 2].

Proposition 2.3. *There exists an invariant mean m on $L^\infty(\mathbb{R}_+)$ such that $m(f_0) = 1$.*

Generally, for a large class of groups it is known that $\mathcal{M}_{\mathrm{t}}(G) \neq \mathcal{M}(G)$. More precisely, if G is a non-compact, non-discrete and locally compact group which is amenable, the sets $\mathcal{M}_{\mathrm{t}}(G)$ and $\mathcal{M}(G)$ are distinct. For details, the reader should see [36, p. 277] and [15], [37], [38].

We will use a description of the set $\mathcal{M}_{\mathrm{t}}(S)$ in the dual space $L^\infty(S)^*$. The counterpart of the result for groups is due to C. Chou [36, p. 138]. For any compact set $K \subseteq S$ of positive measure, we can introduce the mean φ_K on $L^\infty(S)$, defined by $\varphi_K(f) := \frac{1}{|K|}\int_K f(s)\, ds$. Then the next result can be proved in a similar way to the group case [28, Theorem 3, Remark].

Theorem 2.4. *The set $\mathcal{M}_{\rm t}(S)$ is the weak-$*$ closure of the convex hull of the set of all weak-$*$ cluster points of the sequences*

$$\{\varphi_{K_n+s_n}\}_{n\in\mathbb{N}} \quad (\{s_n\}_n \in S^{\mathbb{N}}),$$

where $\{K_n\}_n$ is an arbitrarily given fixed Følner sequence in S°.

2.2. **Types of convergence in semigroups.** Our intention is to prove stability results and discuss the asymptotic properties of operator semigroups. In most cases we shall use weaker notions of the usual convergence of orbits, which requires different types of means. We shall see that these concepts can be described using integral means.

With $\preceq$ we obtain an inductive partial ordering on S, hence S is a directed set. We say that a function $f\colon S \to \mathbb{C}$ tends to 0 at infinity if for every $\varepsilon > 0$ there exists an $s_0 \in S$ such that $|f(s)| < \varepsilon$ whenever $s_0 \preceq s$. Applying invariant means, we can define (strong) almost convergence on semigroups, which is a weaker form of the previous concept of convergence.

Definition 2.5. A function $f \in L^\infty(S)$ is called *almost convergent* if the set $\{m(f) : m \in \mathcal{M}(S)\}$ is a singleton. We shall use the notation a-lim$f = c$ whenever $m(f) = c$ for all $m \in \mathcal{M}(S)$.

Definition 2.6. We say that a function $f \in L^\infty(S)$ *almost converges in the strong sense* to $c \in \mathbb{C}$ if a-lim$|f - c| = 0$.

Now we will introduce a slightly weaker notion of almost convergence. The concept is similar to the previous one, but here we use the set $\mathcal{M}_{\rm t}(S)$ instead of $\mathcal{M}(S)$.

Definition 2.7. A function $f \in L^\infty(S)$ is said to be *topologically almost convergent* if the set $\{m(f) : m \in \mathcal{M}_{\rm t}(S)\}$ is a singleton. We shall use the notation at-lim$f = c$ whenever $m(f) = c$ for all $m \in \mathcal{M}_{\rm t}(S)$. We say that an $f \in L^\infty(S)$ *topologically almost converges in the strong sense* to $c \in \mathbb{C}$ if at-lim $|f - c| = 0$.

The following statement provides us with a clear and simple picture about this type of convergence. We note that its counterpart in $\ell^\infty(\mathbb{Z}_+)$ is the classical characterization of almost convergent sequences due to Lorentz [30].

Proposition 2.8. [28, Proposition 4, Remark] *An $f \in L^\infty(S)$ is topologically almost convergent to c if and only if*

$$\lim_{n\to\infty} \frac{1}{|K_n|}\int_{K_n} f_y(s)\,ds = c$$

uniformly with respect to $y \in S$, where $\{K_n\}_n$ is an arbitrarily chosen Følner sequence.

It is worth mentioning here that if f is an almost convergent function then the above integral condition always holds.

Corollary 2.9. [25, Proposition 7] *If $f \in L^\infty(S)$ is almost convergent with* a-lim $f = c$ *and $\{K_n\}_n$ is a Følner sequence on S, then*

$$\lim_{n\to\infty} \frac{1}{|K_n|}\int_{K_n} f_y(s)\,ds = c$$

uniformly with respect to $y \in S$.

We should add that the converse of the corollary is not true in general due to Proposition 2.3.

After these preliminaries, we can turn to the study of representations having a regular norm-function.

3. Representations with regular norm-function

Let $\mathcal{X}$ be a complex Banach space, and let $\mathcal{L}(\mathcal{X})$ stand for the algebra of bounded linear operators acting on $\mathcal{X}$. A semigroup homomorphism $\rho\colon S \to \mathcal{L}(\mathcal{X})$ is said to be a representation if it is continuous in the strong operator topology; that is,

- $\rho(0) = I$,
- $\rho(s+t) = \rho(s)\rho(t)$ for any $s, t \in S$,
- the orbit $\rho_x \colon S \to \mathcal{X}$, $s \mapsto \rho(s)x$ is continuous for every $x \in \mathcal{X}$.

3.1. **Limit functional and regularity.** Before defining regularity, first we will introduce the gauge function and the limit functional, then provide a summary of their basic properties. We say that the function $p\colon S \to (0,\infty)$ is a *gauge function* if it is measurable and, for every $s \in S$, $p_s/p \in L^\infty(S)$ almost converges in the strong sense to a positive number $c_p(s)$. The function c_p is called the *limit functional* of the gauge function p.

Next we will present an important property of the limit functional, which seems to be crucial for deriving certain results later on.

Lemma 3.1. [25, Lemma 9] *Let p be a gauge function with $p(s) \geq 1$ for $s \in S$. Then $c_p(s) \geq 1$ for every $s \in S$.*

We recall that the non-zero, complex-valued continuous homomorphisms of S are said to be the characters of S. We shall use the notation $S^\sharp$ for the set of characters of S.

Corollary 3.2. [25, Corollary 10] *Let $\chi \in S^\sharp$ be such that $c_p \leq |\chi| \leq p$. Then $|\chi| = c_p$.*

In the following we shall assume that p is bounded on compact sets and that $p \geq 1$ for any gauge p.

We say that the representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ is of *regular norm behaviour with respect to the gauge function p* or has a *p-regular norm-function* if $\|\rho(s)\| \leq p(s)$ holds for every $s \in S$, and $\text{a-lim}_s \|\rho(s)\|/p(s) = 0$ is not true.

The key properties of the limit functional are stated in the following two theorems.

Theorem 3.3. [25, Theorem 13] *Let p be a gauge function on S and let us assume that there exists a representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ with a p-regular norm-function. Then the limit functional c_p of p is a positive character of S.*

Theorem 3.4. [25, Theorem 14] *If the representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ has a regular norm-behaviour with respect to the gauge functions p and q, then*

$$c_p = c_q.$$

The above theorem will lead to the following definition. The function $c_\rho := c_p$ is called the *limit functional of the representation ρ with p-regular norm-function.*

The limit functional and the spectral radius function are related. Similar to discrete semigroups (see [20]), one can prove that the inequality

$$c_\rho(s) \le r(\rho(s)), \quad s \in S,$$

holds for the representation ρ with regular norm-function, where $r(\rho(s))$ denotes the spectral radius of $\rho(s)$. However, it was shown earlier in [19] that for $S = \mathbb{Z}_+$ the limit functional $c_\rho(n)$ is really equal to $r(\rho(n))$ $(n \in \mathbb{Z}_+)$. Now we can state the analogous result concerning C_0-semigroups [25, Proposition 16].

Proposition 3.5. *If the representation* $T\colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ *has a regular norm-behaviour (with respect to a gauge function* p*), then* $c_T(s) = r(T(s))$ $(s \in \mathbb{R}_+)$ *holds.*

Note here that the spectral radius function and the limit functional can be different (see [25, Example 17]).

3.2. **Spectra of representations.** We recall that $C_c(S)$ stands for the set of continuous functions with compact support in S. The Fourier transform of a function $f \in C_c(S)$ with respect to the representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ is given by

$$\widehat{f}(\rho) := \int_S f(s)\rho(s)\, d\mu(s).$$

The integral exists pointwise: $\widehat{f}(\rho)x = \int_S f(s)\rho(s)x\, d\mu(s)$ $(x \in \mathcal{X})$ in the Bochner sense (see e.g. [16, Chapter 7.5]). It is also clear that $\widehat{f}(\rho) \in \mathcal{L}(\mathcal{X})$. We can similarly define $\widehat{f}(\chi)$ when $\chi \in S^\sharp$, since the characters of S are one-dimensional representations.

We shall define the spectrum for unbounded representations related to Lyubich's δ-spectrum [31] and Kérchy's algebraic and balanced spectra [20].

Definition 3.6. The *algebraic spectrum* of the representation ρ is

$$\sigma_{\mathrm{a}}(\rho) := \left\{\, \chi \in S^\sharp : |\widehat{f}(\chi)| \le \|\widehat{f}(\rho)\| \text{ for all } f \in C_c(S) \right\}.$$

The *balanced spectrum* is defined by

$$\sigma_{\mathrm{b}}(\rho) := \sigma_{\mathrm{a}}(\rho) \cap S^\sharp_{\mathrm{b}},$$

where $S^\sharp_{\mathrm{b}} := \{\, \chi \in S^\sharp : \chi(s) \neq 0 \text{ for all } s \in S\}$.
The *spectrum of* ρ with regular norm-function is

$$\sigma(\rho) := \{\chi \in \sigma_{\mathrm{a}}(\rho) : |\chi| \le c_\rho\},$$

where c_ρ denotes the limit functional of ρ.

The existence of the limit functional makes it possible for us to define the peripheral spectrum.

Definition 3.7. The *peripheral spectrum* of the representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ with regular norm-function is defined by

$$\sigma_{\mathrm{per}}(\rho) := \{\, \chi \in \sigma(\rho) : |\chi(s)| = c_\rho(s) \text{ for all } s \in S \,\}.$$

Equipping the set $S^\sharp$ with the compact-open topology, the above spectra form a locally compact, Hausdorff space [25, Proposition 22].

Finally, the point spectrum will be defined. Let the *point spectrum* of the representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ be the set

$$\sigma_{\mathrm{p}}(\rho) := \left\{\chi \in S^\sharp : \text{ there exists } 0 \neq x \in \mathcal{X} \text{ with } \rho(s)x = \chi(s)x \text{ for all } s \in S\right\}.$$

The adjoint $\rho^*(s) := \rho(s)^*$ $(s \in S)$ of ρ is not necessarily strongly continuous, hence the spectrum of ρ^* cannot be defined in general. However, there is no difficulty with defining $\sigma_{\mathrm{p}}(\rho^*)$ in an analogous way to $\sigma_{\mathrm{p}}(\rho)$.

One of our results is connected to the balanced spectrum [25, Proposition 19].

Proposition 3.8. *If $\rho\colon S \to \mathcal{L}(\mathcal{X})$ is a representation with regular norm-function, then $\sigma_{\mathrm{b}}(\rho) \subseteq \sigma(\rho)$.*

If $S = \mathbb{Z}_+^n$ or $\mathbb{R}_+^n$, then it can be easily checked that every character of S is non-vanishing; thus $\sigma_{\mathrm{a}}(\rho)$, $\sigma_{\mathrm{b}}(\rho)$ coincide and they are equal to $\sigma(\rho)$ when ρ has a regular norm-function.

3.3. **Description of spectra.** (a) Let T be a bounded, linear operator on $\mathcal{X}$. We shall denote the representation induced by T by $\rho_T\colon \mathbb{Z}_+ \to \mathcal{L}(\mathcal{X})$. Then it can be shown that $\sigma_{\mathrm{a}}(\rho_T) = \sigma_{\mathrm{b}}(\rho_T) = \widehat{\sigma(T)}$, where $\widehat{\sigma(T)}$ denotes the polynomially convex hull of $\sigma(T)$, the spectrum of T [35, Theorem 2.10.3].

(b) A geometrically similar result can be proved for the representations of $\mathbb{R}_+$; that is, for C_0-semigroups [28, Proposition 5 and Corollary 6]. Let $T\colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ be a C_0-semigroup having the generator A, and let $\rho_\infty(A)$ be the component of $\mathbb{C} \setminus \sigma(A)$ which contains the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > \omega_0(T)\}$, where $\omega_0(T) := \lim_{t\to\infty}(\log\|T(t)\|)/t$.

Theorem 3.9. *Using the above notations, we have*

$$\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{b}}(T) = \mathbb{C} \setminus \rho_\infty(A).$$

(c) Let us assume that the representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ is bounded: $\alpha := \sup\{\|\rho(s)\| : s \in S\} < \infty$. If $\|\rho(s_0)\| < 1$ holds for some $s_0 \in S$, then the inequalities $\|\rho(ns_0 + s)\| \le \|\rho(s_0)\|^n \alpha$ $(n \in \mathbb{N})$ tell us that $\lim_s \|\rho(s)\| = 0$, i.e. ρ is uniformly stable.

Assuming that $\|\rho(s)\| \ge 1$ is true for every $s \in S$, we can readily prove that ρ is of regular norm behaviour with respect to the gauge function $p(s) := \alpha$ $(s \in S)$. The limit functional c_ρ of ρ is clearly the constant $\mathbf{1}$ function. Thus $\sigma_{\mathrm{per}}(\rho)$ coincides with the *unitary spectrum* $\sigma_{\mathrm{u}}(\rho) := \{\chi \in \sigma(\rho) : |\chi| = 1\}$ of ρ and it can be easily verified that $\sigma_{\mathrm{a}}(\rho) = \sigma(\rho)$ is also true.

Taking into account the fact that $C_c(S)$ forms a dense subset of $L^1(S)$, we conclude that if $|\widehat{f}(\chi)| \le \|\widehat{f}(\rho)\|$ holds for every $f \in C_c(S)$ then it does so for every $f \in L^1(S)$ too. Thus $\sigma(\rho)$ coincides with the spectrum introduced by Batty and Vũ for bounded representations in [5]. We recall that this concept is an adaptation of the finite L-spectrum and the Arveson spectrum, defined for group representations, to the semigroup setting (see [31] and [10]).

(d) Let $\rho\colon S \to \mathcal{L}(\mathcal{X})$ be a representation of regular norm-behaviour. Since $c_\rho \in S_b^\sharp$, the representation $\tilde{\rho} := c_\rho^{-1}\rho\colon S \to \mathcal{L}(\mathcal{X})$ is also of regular norm-behaviour and $c_{\tilde{\rho}} = 1$. Obviously every $\chi \in \sigma_{\mathrm{per}}(\tilde{\rho})$ can be uniquely extended to a character $\tilde{\chi}$ of the extension group G. We conclude that $\sigma_{\mathrm{per}}(\tilde{\rho})$ may be identified with the unitary spectrum $\mathrm{Sp}_{\mathrm{u}}(\tilde{\rho})$ introduced in [3], namely $\sigma_{\mathrm{per}}(\tilde{\rho}) = \{\tilde{\chi}|S : \tilde{\chi} \in \mathrm{Sp}_{\mathrm{u}}(\tilde{\rho})\}$. Therefore $\sigma_{\mathrm{per}}(\rho) = \{c_\rho(\tilde{\chi}|S) : \tilde{\chi} \in \mathrm{Sp}_{\mathrm{u}}(\tilde{\rho})\}$ is true.

4. The stability theorem

4.1. **Regularity and isometric representations.** One important part of the proof of the stability theorem is that we can associate an isometric representation with the original one. The statement is well known for bounded representations and we can extend the result under the regularity condition [25, Theorem 23].

Theorem 4.1. *For any representation $\rho\colon S \to \mathcal{L}(\mathcal{X})$ with p-regular norm-function, there exists an isometric representation $\psi\colon S \to \mathcal{L}(\mathcal{Y})$ on a Banach space $\mathcal{Y}$ and a contractive transformation $Q \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that:*

(i) $\ker Q = \{x \in \mathcal{X} : \text{a-}\lim_s \|\rho(s)x\|/p(s) = 0\}$, *and* $\operatorname{ran} Q$ *is dense in* $\mathcal{Y}$;
(ii) $Q\rho(s) = c_\rho(s)\,\psi(s)Q$ *holds for every* $s \in S$;
(iii) *for every operator* $C \in \{\rho(S)\}'$, *there exists a unique operator* $D \in \{\psi(S)\}'$ *such that* $QC = DQ$; *furthermore, the mapping* $\gamma\colon \{\rho(S)\}' \mapsto \{\psi(S)\}', C \mapsto D$ *is a contractive algebra-homomorphism;*
(iv) $\sigma(\rho) \supseteq c_\rho\sigma(\psi), \quad \sigma_{\mathrm{per}}(\rho) \supseteq c_\rho\sigma_{\mathrm{per}}(\psi), \quad \sigma_{\mathrm{p}}(\rho^*) \supseteq c_\rho\sigma_{\mathrm{p}}(\psi^*)$.

4.2. **The stability theorem.** The next statement, the generalization of the Arendt–Batty–Lyubich–Vũ theorem for representations that have a regular norm-function [25, Theorem 25], is one of the main results of our thesis.

Theorem 4.2. *Let $\rho\colon S \to \mathcal{L}(\mathcal{X})$ be a representation with a p-regular norm-function. If $\sigma_{\mathrm{per}}(\rho)$ is countable and $\sigma_{\mathrm{p}}(\rho^*) \cap \{\chi \in S^\sharp : |\chi| = c_\rho\}$ is empty, then*

$$\text{a-}\lim_s \frac{\|\rho(s)x\|}{p(s)} = 0$$

holds for all $x \in \mathcal{X}$.

Applying Corollary 2.9 we obtain the following [25, Corollary 26].

Corollary 4.3. *Let $\rho\colon S \to \mathcal{L}(\mathcal{X})$ be a representation with a p-regular norm-function. If $\sigma_{\mathrm{per}}(\rho)$ is countable and $\sigma_{\mathrm{p}}(\rho^*) \cap \{\chi \in S^\sharp : |\chi| = c_\rho\}$ is empty, then*

$$\lim_{i\to\infty} \frac{1}{\mu(K_i)} \int_{K_i} \frac{\|\rho(s)x\|}{p(s)}\, d\mu(s) = 0$$

is true for all $x \in \mathcal{X}$, where $\{K_i\}_i$ is any Følner sequence.

The previous result is a generalization of the stability result [5, Theorem 4.2] concerning bounded representations. The spectral conditions of Theorem 4.2 are essentially the same as those in the main result Theorem 3.2 of [3]. The differences lie in the norm-condition on ρ and in the nature of convergence of orbits.

4.3. **Representations and topological regularity.** Here it is important to notice that we use the set of invariant means to define regularity, but the previous results remain valid if we just apply the set of topologically invariant means. (In fact, the main ingredients like the limit functional and the associated isometric representation can be introduced in a similar way with $\mathcal{M}_{\mathrm{t}}(S)$ instead of $\mathcal{M}(S)$.) For discrete semigroups these two concepts are the same because the classes of these means coincide in the discrete case. In the next section we will present a detailed study on the second alternative on the real half-line, introducing C_0-semigroups with a topologically regular norm-function.

5. C_0-semigroups and topological regularity

Topological regularity is defined in the following way. We say that $p\colon \mathbb{R}_+ \to [1, \infty)$ is a *topological gauge function* if (i) it is measurable, (ii) for every $s \in \mathbb{R}_+$, $p_s/p \in L^\infty(\mathbb{R}_+)$ topologically almost converges in the strong sense to a positive real number $c_p(s)$, and (iii) the functions p and $\psi(s) = \sup_{t\in\mathbb{R}_+} p_s(t)/p(t)$ are locally bounded (i.e. bounded on compact sets). The function c_p is called the *limit functional* of the gauge function p. The set of topological gauge functions shall be denoted by $\mathcal{P}_{\mathrm{t}}$. In the following, a representation of $\mathbb{R}_+$ is called a C_0-semigroup.

Definition 5.1. The C_0-semigroup $T\colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ has *regular norm behaviour with respect to the topological gauge function* p or has a *p-regular norm-function* if (i) $\|T(s)\| \le p(s)$ holds for every $s \in \mathbb{R}_+$, and (ii) at-$\lim_s \|T(s)\|/p(s) = 0$ does not hold.

Operators with regular norm-sequences were characterized by L. Kérchy and V. Müller in [26]. Following the discrete case, let us introduce the *regularity constant* κ_T for a C_0-semigroup T if $r(T(s)) > 0$ for some (and then for all) $s \in \mathbb{R}_+$. Then

$$\kappa_T := \inf_{n\in\mathbb{N}} \sup_{s\in\mathbb{R}_+} \left[\left(\frac{1}{n} \int_s^{s+n} r(T(t))^{-1} \|T(t)\| \, dt \right) \left(\sup_{s\le y\le s+n} r(T(y))^{-1} \|T(y)\| \right)^{-1} \right].$$

Clearly, we have $0 \le \kappa_T \le 1$. The regularity constant makes it possible for us to give a description of a semigroup whose norm-function exhibits a regular behaviour.

The next result is proved in [28, Theorem 8].

Theorem 5.2. *Let $T\colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{X})$ be a C_0-semigroup. Then the following conditions are equivalent:*

(i) *T has a p-regular norm-function with a topological gauge function $p \in \mathcal{P}_{\mathrm{t}}$,*
(ii) *T has a p-regular norm-function with a continuous gauge $p \in \mathcal{P}_{\mathrm{t}}$,*
(iii) *$\|T(s)\| \ge 1$ for every $s \in \mathbb{R}_+$ and $\kappa_T > 0$.*

6. A Katznelson–Tzafriri type theorem in Hilbert spaces

The Katznelson–Tzafriri theorem is an operator-theoretic result which is related to the ABLV theorem. (To learn more about the connection, the reader is asked to consult [13].) When $S = \mathbb{Z}_+$, we can present an extension of the result in the Hilbert space setting.

Let I be the identity operator on $\mathcal{X}$. If $f \in A^+(\mathbb{T})$ and T is power-bounded operator, then a bounded functional calculus naturally arises which can be defined by $f(T) := \sum_{k=0}^{\infty} \widehat{f}(k) T^k \in \mathcal{L}(\mathcal{X})$, where $f(\lambda) = \sum_{k=0}^{\infty} \widehat{f}(k)\lambda^k$ and $\sum_{k=0}^{\infty} |\widehat{f}(k)| < \infty$.

Our starting point is an observation which leads us to introduce the ergodic condition in our generalization [29, Lemma 2.2].

Lemma 6.1. *Let T be a power-bounded operator on a complex Banach space $\mathcal{X}$ and let $f \in A^+(\mathbb{T})$. Then, for every $\lambda \in \mathbb{T}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k \left(f(T) - f(\lambda) I \right) \right\| = 0.$$

The uniform ergodic theorem tells us that $\frac{1}{n}\sum_{k=0}^{n-1} T^k$ tends to zero in norm if and only if 1 is in the resolvent set of T (cf. [27, Theorem 2.7]). With this result, a simple corollary of the above lemma is straightforward [29, Corollary 2.3].

Corollary 6.2. *Let T be a power-bounded operator on a Banach space $\mathcal{X}$ and $f \in A^+(\mathbb{T})$. Then, for each $\lambda \in \sigma(T) \cap \mathbb{T}$,*

$$f(\lambda) = 0 \text{ if and only if } \lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k f(T) \right\| = 0.$$

Our main result will now be presented [29, Theorem 2.1].

Theorem 6.3. *Let T be a power-bounded operator on a Hilbert space $\mathcal{H}$. If $Q \in \mathcal{L}(\mathcal{H})$ and $TQ = QT$, then the following statements are equivalent:*

(i) $\lim_{n\to\infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \lambda^{-k} T^k Q \right\| = 0$ *for every* $\lambda \in \sigma(T) \cap \mathbb{T}$,

(ii) $\lim_{n\to\infty} \|T^n Q\| = 0$.

Moreover, if $Q = f(T)$ for some $f \in A^+(\mathbb{T})$, then (i) *and* (ii) *are equivalent to*

(iii) $f(\lambda) = 0$ *for every* $\lambda \in \sigma(T) \cap \mathbb{T}$.

The proof partly follows Vũ's method ([39], [40]); that is, we first verify convergence in the strong operator topology by reducing the problem to isometries. After that we can complete the proof using some aspects of an ultrapower approach.

It is an open problem whether one can prove a similar statement for C_0-semigroups or more general representations. It is also an open question whether the statement remains valid in more general spaces; for instance, in L^p-spaces ($1 < p < \infty$) or in superreflexive Banach spaces.

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