Szemistabilis eloszlások geometriai parciális vonzástartományából vett véletlen változók lineáris kombinációi

Doktori értekezés

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Introduction

The theory of limit theorems is a classical and important part of probability theory. The first of these theorems was proved by de Moivre in 1733 for independent Bernoulli trials. This was reinvented by Laplace in 1812. The method of characteristic functions, which can handle the problem in its full generality, was born only at the beginning of the 20^{th} century. Since then, an outblasting development started in the theory, due to such excellent mathematicians as Ljapunov, Lévy, Lindeberg, Khinchin, Kolmogorov, Doeblin, Gnedenko and Feller.

For a long time the only method for dealing with sums of independent variables was Fourier analysis, which obscures the underlying probabilistic intuition. Therefore the *probabilistic approach* of Csörgő, Haeusler and Mason [18] was a milestone in the theory. This approach is based upon the asymptotic behavior of the uniform empirical distribution function in conjunction with the tail probabilities of the corresponding quantile function. The method is applicable only when the independent random variables are also identically distributed. However, in this case the probabilistic approach is capable to handle lightly trimmed sums, that is when some of the largest and some of the smallest summands are eliminated.

The semistable distributions play central role in this work. The notion first appears in Lévy's [30] classical work as a natural generalization of the stable laws. Later, due to the works of Kruglov [29] and Mejzler [33], the definition changed a bit, and the importance of semistable laws turned out. Stable distributions arise as limiting distributions of suitably centralized and normalized sums of iid random variables, along the whole sequence $\mathbb{N}$. According to a famous theorem of Khinchin, when the convergence is not demanded along the whole sequence of natural numbers only through a subsequence, then every infinitely divisible distribution can be reached. If there is a geometric growth condition on the subsequence, we get an intermediate class, the class of semistable distributions. To be more precise (the equivalent version of) the definition is the following:

Non-degenerate distributions that arise as limiting distributions of suit-

ably centralized and normalized sums of iid random variables along subsequences $\{k_n\}_{n=1}^{\infty}$ *satisfying* $k_{n+1}/k_n \to c$*, for some* $c \geq 1$ *are called* semistable laws.

Megyesi [32] applied the probabilistic approach for semistable distributions, and he characterized the geometric partial attraction.

In the most general setup of limit theorems, for row-wise independent random variables, there are necessary and sufficient conditions for the existence of the limit distribution. The case of iid random variables is also examined in detail. The natural assumption between the two cases, that is the case of linear combinations of iid random variables is less analyzed. This is the subject of the current theses.

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables from the domain of geometric partial attraction of a semistable law. These random variables can be viewed as the gains in ducats (losses when negative) in an independent repetitions of a game of chance. We assume that Peter, the banker plays exactly one game with each of the n gamblers, Paul$_1$, Paul$_2$, …, Paul$_n$. Before they play our Pauls may agree to use a pooling strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$, in which the components are non-negative, and add to unity. Under this strategy Paul$_1$ receives $p_{1,n}X_1 + p_{2,n}X_2 + \cdots + p_{n,n}X_n$, Paul$_2$ receives $p_{n,n}X_1 + p_{1,n}X_2 + \cdots + p_{n-1,n}X_n$, Paul$_3$ receives $p_{n-1,n}X_1 + p_{n,n}X_2 + p_{1,n}X_3 + \cdots + p_{n-2,n}X_n$, …, and Paul$_n$ receives $p_{2,n}X_1 + p_{3,n}X_2 + \cdots + p_{n,n}X_{n-1} + p_{1,n}X_n$ ducats. In the theses we investigate the properties of the weighted sum $p_{1,n}X_1 + p_{2,n}X_2 + \cdots + p_{n,n}X_n$, which in our context is the gain of Paul$_1$ with strategy $\boldsymbol{p}_n$.

In Chapter 1 we consider a very special case, when the game of chance is the generalized St. Petersburg(p) game. In this game Peter tosses a possibly biased coin, until it lands 'heads', and pays r^k ducats to Paul, if this happens on the k^{th} toss, that is the distribution of Paul's gain X is $\mathbf{P}\{X = r^k\} = q^{k-1}p$, where $r = 1/q$, $q = 1 - p$, and $p \in (0, 1)$ is the probability of 'heads' at each throw. The classical version of the game, which is played with a fair coin, is due to Nicolaus Bernoulli from 1713. The original problem was the fair price for n game. Once this price is agreed, it is wholly indifferent to Peter whether the other side is our old Paul playing n games in a row, or a company of n gamblers, Paul$_1$, Paul$_2$, …, Paul$_n$, each playing exactly one game with respective individual winnings $X_1, X_2, \ldots, X_n$, and cooperating among themselves. However, it turns out that there are strategies, so called *admissible strategies*, which are better to every Paul, in the sense that each of them receives more ducats, than with the individualistic strategy. The admissible strategies play significant role in the next two chapters. These paradoxical results for mutually beneficial sharing among any fixed number of classical St. Petersburg gamblers were obtained by Csörgő and Simons.

The extension to the general case is not straightforward because, unlike in the classical case with $p = 1/2$, admissibly pooled winnings generally fail to stochastically dominate individual ones for more than two gamblers. Best admissible pooling strategies are determined when p is rational, and the algebraic depth of the problem for an irrational p is illustrated by an example.

We emphasize that in the whole chapter n, the number of Pauls will be fixed, does not tend to infinity. Therefore here we do not deal with asymptotic results. Thus the paradoxical results come from only the fact that the expectation is infinite. The proofs are mainly elementary, the only deeper tool we use, is the comparison operator, which allows us to compare random variables with infinite mean.

This kind paradox was the motivation for the author for further investigation of linear combinations. This part is the extended version of [26].

In Chapter 2 we determine the asymptotic behavior of the gain of Paul_1, in a more general St. Petersburg(α, p) game, where $\alpha \in (0, 2)$. In this game the distribution of Paul's winning X is $\mathbf{P}\{X = r^{k/\alpha}\} = q^{k-1}p$. That is we are still dealing with a specific distribution from the domain of geometric partial attraction of a specific semistable law. However, this special case allows us to prove merging asymptotic expansions for the gains, which cannot be hoped for the general case. For an arbitrary strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ we define the random variable $S_{\boldsymbol{p}_n}^{\alpha,p} = p_{1,n}^{1/\alpha} X_1 + p_{2,n}^{1/\alpha} X_2 + \cdots + p_{n,n}^{1/\alpha} X_n - p/q H_\alpha(\boldsymbol{p}_n)$, where $H_\alpha(\boldsymbol{p}_n)$ depends only on the strategy. The main results are short asymptotic expansions, given in terms of Fourier – Stieltjes transforms and are constructed from suitably chosen members of the classes of subsequential semistable infinitely divisible asymptotic distributions for the total winnings of the n players and from their pooling strategy, where the classes themselves are determined by the two parameters of the game. For all values of the tail parameter the expansions yield best possible rates of uniform merge. In the general case there is no hope for limit theorem, since by the Doeblin – Gnedenko criterion it does not exist even for the uniform strategy. Surprisingly, it turns out that for a subclass of strategies not containing the averaging uniform strategy, but containing the admissible ones, our merging approximations reduce to asymptotic expansions of the usual type derived from a proper limiting distribution. The Fourier – Stieltjes transforms are shown to be numerically invertible in general and it is also demonstrated that the merging expansions provide excellent approximations even for very small n.

We use Fourier analytic methods. One of the main tools is the Esseen-inequality, which gives upper bound for the supremum distance of functions in terms of the difference of their Fourier – Stieltjes transforms. The other important thing is the existence of the mixed derivatives of semistable distribution functions, which was derived by Csörgő in [10] for the St. Petersburg case, and in [9] in general. This chapter contains the results of [12].

In Chapter 3 we investigate the asymptotic distribution of the gain in its whole generality. Let $X_1, X_2, \ldots$ be iid random variables from the domain of geometric partial attraction of a semistable law, with characteristic exponent $\alpha \in (0,2)$. Consider the pooling strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$, and define the random variable $S_{\alpha,\boldsymbol{p}_n} = p_{1,n}^{1/\alpha} X_1/\ell(p_{1,n}) + p_{2,n}^{1/\alpha} X_2/\ell(p_{2,n}) + \cdots + p_{n,n}^{1/\alpha} X_n/\ell(p_{n,n}) - \mu(\boldsymbol{p}_n)$, where $\ell(\cdot)$ is a slowly varying function at 0. We prove merge theorems along the entire sequence of natural numbers for the distribution functions of $S_{\alpha,\boldsymbol{p}_n}$. For some sequences of linear combinations, not too far from those with equal weights, the merge theorems reduce to ordinary asymptotic distributions with semistable limits. This result finally lightens the importance of the admissible strategies.

The first merge theorems were obtained by Csörgő [7] and there are some merge theorems in Chapter 2 also. However, in all cases merge meant that the supremum distance of the corresponding distribution functions tends to 0, with an appropriate rate. Therefore in these cases Esseen's inequality did the job, and there were no need to work out general conditions of merge. In our case, with no more assumption on the underlying distribution function, there is no hope for rates. We give a general definition for merge through the Lévy metric, and we obtain sufficient conditions in terms of characteristic functions. We note that D'Aristotile, Diaconis and Freedman [19] investigated five different definitions of merge in a separable metric space. The most important case, the case of real random variables, is passed over, and they do not analyze the relationship of (any notion of) merge with the characteristic functions. Beside of merge the most important tool in this chapter is the probabilistic approach of Csörgő, Haeusler and Mason [18]. This part is taken from [28].

Chapter 4 makes the whole story round. We are dealing with the asymptotic normality of an arbitrary linear combination of iid variables. Let $X_1, X_2, \ldots$ be iid random variables, and consider $\boldsymbol{a}_n = (a_{1,n}, \ldots, a_{n,n})$, an arbitrary sequence of weights. We investigate the asymptotic normality of the sum $S_{\boldsymbol{a}_n} = a_{1,n}X_1 + \cdots + a_{n,n}X_n$ under the natural negligibility condition $\lim_{n\to\infty} \max\{|a_{k,n}| : k = 1, \ldots, n\} = 0$. In the finite variance case we prove necessary and sufficient condition for the distributional convergence. We also show that if $S_{\boldsymbol{a}_n}$ is asymptotically normal for a weight sequence $\boldsymbol{a}_n$, in which the components are of the same magnitude, then the common distribution belongs to the attraction of the normal law. Here we use the classical theory of sum of independent random variables.

Chapter 1.

Motivation: The generalized n-Paul paradox

1.1. The St. Petersburg paradox

Peter offers to let Paul toss a fair coin until it lands heads and pays him 2^k ducats if this happens on the k^{th} toss. What is the price for Paul to make the game 'equal and fair'? If X denotes Paul's gain, then

$$\mathbf{P}\{X = 2^k\} = \frac{1}{2^k} \quad \text{thus} \quad \mathbf{E}(X) = \sum_{k=1}^{\infty} 2^k \, \mathbf{P}\{X = 2^k\} = \sum_{k=1}^{\infty} 1 = \infty,$$

so the price of the game is infinite number of ducats. However as Nicolaus Bernoulli wrote, 'there ought not be a sane man who would not happily sell his chance for forty ducats'. This is the famous St. Petersburg paradox, which was first raised by Nicolaus Bernoulli [3] in 1713, in a letter to de Montmort. The problem appears in the book *Essay d'Analyse sur les Jeux de Hazard* by Pierre Rémond de Montmort in 1713, in its original form, played with dice. Gabriel Cramer learned about the paradox in 1726, and he was, who simplified the problem to its coin-tossing version. At that time the central problem of probability theory was the 'fair price', the equitas, therefore this phenomena of infinite expectation was so unexplainable. However, as Feller [20] writes, 'the modern student will hardly understand the mysterious discussions of this paradox.' We delineate some trials from the numerous 'resolutions'. Gabriel Cramer cut the possible gain at 2^k, for some large k. He explained that for some large integer k, a common man's happiness for the gain of 2^k ducats is the same, as for any larger number of ducats. So the value of a game is

$$\sum_{j=1}^{k} 2^j \frac{1}{2^j} + \sum_{j=k+1}^{\infty} 2^k \frac{1}{2^j} = k + \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j = k+1$$

ducats. But what is k? The younger cousin of Nicolaus, Daniel Bernoulli had a similar theory: the more money Paul has, the less is his happiness for a fix number of ducats. This implies that the fair price of the game would depend on the wealth of the gambler. With such moral treatments neither of them get closer to the solution. The problem became so popular that it occurs in the greatest work of the french enlightenment, in the Encyclopédie: *Croix ou pile*, that is Heads or Tails is written by Jean le Rond d'Alembert. For almost 200 years the greatest mathematicians (Euler, Lagrange, Laplace, Poisson) did not get any advance in the problem. At the same time the endless disputations about the paradox greatly promoted the utilitarian economy, which is started by Daniel Bernoulli's thoughts [2].

For a single game, everything can be known for the gain X from its distribution function. One can check easily that it is

$$F(x) = \mathbf{P}\{X \le x\} = \begin{cases} 0, & \text{if } x < 2\,, \\ 1 - \frac{1}{2^{\lfloor \log_2 x \rfloor}} = 1 - \frac{2^{\langle \log_2 x \rangle}}{x}, & \text{if } x \ge 2\,, \end{cases}$$

where $\lfloor y \rfloor$ stands for the (lower) integer part of y, while $\langle y \rangle$ is the fractional part and $\log_r$ is the base r logarithm. Of course the real question is how much should Paul pay for n game, in which his total gain is $S_n = X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \ldots, X_n$ are independent St. Petersburg random variables. The first mathematically explicit result concerning the paradox is due to Feller [20] in 1945. He proved the following weak law:

$$\frac{S_n}{n \log_2 n} \xrightarrow{\mathbf{P}} 1\,,$$

where $\xrightarrow{\mathbf{P}}$ stands for the convergence in probability, and $n \to \infty$. (Throughout the theses, an asymptotic relationship is meant as $n \to \infty$ unless otherwise specified.) That is according to Feller the fair price for n games is $n \log_2 n$ ducats. However it turned out that this is a very nice example when the strong version of the weak law is not true. Namely the stochastic limit is the almost sure lim inf, so for Peter, the banker $n \log_2 n$ ducats is not enough! Chow and Robbins [5] and Adler [1] proved that

$$\limsup_{n\to\infty} \frac{S_n}{n \log_2 n} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{S_n}{n \log_2 n} = 1$$

almost surely. (Actually Chow and Robbins show in general that if the expectation is infinite then no strong theorems hold.) For the deeper reasons of the phenomena we refer to [14]. Moreover Csörgő [7] explains why the fair price for S_n cannot be determined by laws of large numbers.

The non-existence of the strong laws of large numbers does not exhaust the curiosity of the paradox. To be honest, enough has left, as we will see soon. In the distribution function the numerator $2^{\langle \log_2 x \rangle}$ is not slowly varying at infinity. Therefore the classical Doeblin – Gnedenko criterion implies that there is no limit theorem for the sums. The limit theorems are the topic of the next two chapters. For further historical background we refer to [6].

1.2. The n-Paul problem

Peter agrees with two player, Paul_1 and Paul_2, that he plays exactly one St. Petersburg game with each of them. Our players may decide to keep their own winnings, or before they play they agree to share the gain, each of them receiving $(X_1 + X_2)/2$ ducats. Which is the better strategy? Of course if the expectation was finite, then these strategies would be equally good, neither is superior. But now, as we will see soon, the averaging strategy is better. Despite Peter pays out the same amount of ducats, each of our Pauls get in the average one extra ducat! This is the two-Paul paradox. Indeed, let $S_2 = X_1 + X_2$ and $U_2 = 2X_1 + X_2 I\{X_2 \le X_1\}$, where $I\{A\}$ is the indicator function of the event A. Csörgő and Simons [15] proved the distributional equality

$$S_2 \overset{\mathcal{D}}{=} U_2.$$

With this equality we can determine how much does the averaging strategy better than the individualistic. Following [15] we have

$$\mathbf{P}\big\{X_1 \ge 2^k \,|\, X_2 = 2^k\big\} = \mathbf{P}\big\{X_1 \ge 2^k\big\} = 1 - \sum_{j=1}^{k-1} \left(\frac{1}{2}\right)^j = 1 - \left[1 - \frac{1}{2^{k-1}}\right] = \frac{2}{2^k}$$

for all $k \in \mathbb{N}$, therefore $\mathbf{P}\{X_1 \ge X_2 \,|\, X_2\} = 2/X_2$. So we obtain that

$$\mathbf{E}\big(X_2\, I\{X_2 \le X_1\}\big) = \mathbf{E}\big(X_2\, \mathbf{P}\{X_1 \ge X_2 \,|\, X_2\}\big) = \mathbf{E}\left(X_2\, \frac{2}{X_2}\right) = 2,$$

that is the averaging strategy implies one (2/2) extra ducat for both Pauls.

Actually, we compared two random variables with infinite expectation. Now it is natural to define the *comparison operator* (if it exists) of the random variables U and V as:

$$\mathbf{E}[U,V]=\int_{-\infty}^{\infty}\big[\mathbf{P}\{U>x\}-\mathbf{P}\{V>x\}\big]\mathrm{d}x.$$

We refer to Csörgő and Simons [15], [17] for a detailed exposition and discussion of the comparison operator $\mathbf{E}[\cdot\,,\cdot]$. We note that if the integral exists as a Lebesgue integral, then it exists in the Riemann sense, while the converse is not true. (In the followings the phrase Riemann sense is always meant as improper Riemann sense.) For an example see [15]. Moreover $\mathbf{E}[U,V]=\mathbf{E}(U)-\mathbf{E}(V)$, whenever $\mathbf{E}(U)$ and $\mathbf{E}(V)$ exist and at least one of them is finite. (We use the usual convention that $\pm\infty-c=\pm\infty=c-\mp\infty$.)

With these notations in the 2-Paul case we obtained $\mathbf{E}[X_1+X_2,2X_1]=2$, thus $\mathbf{E}[(X_1+X_2)/2,X_1]=1$. Csörgő and Simons also investigated the case, when $n=2^k$ Pauls play one game with Peter. Then $\mathbf{E}[S_{2^k}/2^k,X_1]=k$, for each k, where $X_1,\ldots,X_n$ are independent St. Petersburg games, and S_n is its sum. The 3-Paul case is much more complicated. Assume that Peter plays exactly one game with Paul$_1$, Paul$_2$ and Paul$_3$, and their winnings are X_1, X_2 and X_3 respectively. It seems plausible that the averaging strategy, giving each Paul $S_3/3$ ducats, dominates the individualistic one. However, it turns out that the two strategy are not comparable, that is the integral

$$\mathbf{E}\left[\frac{S_3}{3},X_1\right]=\int_0^{\infty}\left[\mathbf{P}\left\{\frac{S_3}{3}>x\right\}-\mathbf{P}\{X_1>x\}\right]\mathrm{d}x$$

does not exist, even in the improper Riemann sense. This immediately implies that the variables X_1 and $S_3/3$ are stochastically incomparable. Indeed $\mathbf{P}\{X_1=2\}=\frac{1}{2}$ and $\mathbf{P}\big\{S_3/3=2\big\}=\frac{1}{8}<\frac{1}{2}$, while $\mathbf{P}\{X_1<8\}=\frac{3}{4}$ and $\mathbf{P}\big\{S_3/3<8\big\}=0.76171875>\frac{3}{4}$.

But there are two other pooling strategies for the 3-Paul case investigated by Csörgő and Simons [15]. The simpler is that each Paul gives his winning to the other two, half to each. The 2-Paul case implies that this strategy provides one extra ducat for each of the three Pauls. The second strategy is more interesting: Paul$_1$ ends up with $\frac{1}{2}X_1+\frac{1}{4}X_2+\frac{1}{4}X_3$, Paul$_2$ with $\frac{1}{2}X_2+\frac{1}{4}X_3+\frac{1}{4}X_1$, and Paul$_3$ with $\frac{1}{2}X_3+\frac{1}{4}X_1+\frac{1}{4}X_2$. This strategy provides one and a half extra ducats for each of the three Pauls.

Before playing with more Pauls, we generalize the problem.

Peter offers to let Paul toss a **possibly biased coin** until it lands heads and pays him r^k ducats if this happens on the k^{th} toss, $k\in\mathbb{N}=\{1,2,\ldots\}$, where $r=1/q$ for $q=1-p$ and $p\in(0,1)$ is the probability of 'heads' at each throw. This is the generalized St. Petersburg(p) game, in which $\mathbf{P}\{X=r^k\}=q^{k-1}p$, $k\in\mathbb{N}$, for Paul's gain X. Clearly, we get the classical version if $p = 1/2$. The distribution function now

$$F_p(x) = \mathbf{P}\{X \le x\} = \begin{cases} 0, & \text{if } x < r\,, \\ 1 - q^{\lfloor \log_r x \rfloor} = 1 - \frac{r^{\langle \log_r x \rangle}}{x}, & \text{if } x \ge r\,. \end{cases}$$

Let $X_1, X_2, \ldots$ be independent St. Petersburg(p) variables and S_n its sum. Then Feller's weak law has the following general form

$$\frac{S_n}{n \log_r n} \xrightarrow{\mathbf{P}} \frac{p}{q}.$$

Return to the problem of our cooperating Pauls, whose number n is now arbitrary. Suppose that Peter agrees to play exactly one St. Petersburg(p) game with a company of n gamblers, Paul$_1$, Paul$_2$, ..., Paul$_n$, whose respective individual winnings $X_1, X_2, \ldots, X_n$. They may agree to use a *pooling strategy* $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$, where $p_{1,n}, \ldots, p_{n,n} \ge 0$ and $\sum_{k=1}^n p_{k,n} = 1$. Under this strategy, Paul$_1$ is to receive the amount $p_{1,n}X_1 + p_{2,n}X_2 + \cdots + p_{n,n}X_n$, Paul$_2$ is to receive the amount $p_{n,n}X_1 + p_{1,n}X_2 + \cdots + p_{n-1,n}X_n$, Paul$_3$ is to receive the amount $p_{n-1,n}X_1 + p_{n,n}X_2 + p_{1,n}X_3 + \cdots + p_{n-2,n}X_n$, ..., and Paul$_n$ is to receive the amount of $p_{2,n}X_1 + p_{3,n}X_2 + \cdots + p_{n,n}X_{n-1} + p_{1,n}X_n$ ducats. Under these rotating assignments of weights, every bit of all of the individual winnings is paid out. Moreover, this strategy is fair to every Paul in the sense that their winnings are equally distributed and each receives the same *added value* equal to

$$\begin{aligned} A_p(\boldsymbol{p}_n) &= \mathbf{E}[\,p_{1,n}X_1 + \cdots + p_{n,n}X_n, X_1] \\ &= \int_0^\infty \big[\mathbf{P}\{p_{1,n}X_1 + \cdots + p_{n,n}X_n > x\} - \mathbf{P}\{X_1 > x\}\big]\,\mathrm{d}x, \end{aligned} \tag{1.1}$$

whenever the integral is defined, so that comparison is possible.

The aim of this chapter is to determine those strategies, so called *admissible strategies*, for which the added value $A_p(\boldsymbol{p}_n)$ exist, and we discuss the manner of the existence (improper Riemann, or Lebesgue). If the parameter p of the game is rational we determine the best admissible strategies, which yield the greatest added value to our Pauls, and in the irrational case we point out the algebraic depth of the problem by an example. These results are the generalizations of the results of Csörgő and Simons [17].

1.3. Results and discussion

We call a strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ *admissible* if each of its components is either zero or a nonnegative integer power of $q = 1 - p$. Individualistic strategies $(1, 0, \dots, 0)$ are thus admissible for each p, otherwise the powers in nonzero components are positive integers. The entropy of a pooling strategy is $H_r(\boldsymbol{p}_n) = \sum_{j=1}^n p_{j,n} \log_r 1/p_{j,n}$, where $\log_r$ denotes the base r logarithm and $0 \log_r 1/0 = 0$. We say that the random variable U is stochastically larger than the random variable V, written $U \geq_{\mathcal{D}} V$, if $\mathbf{P}\{U > x\} \geq \mathbf{P}\{V > x\}$ for all $x \in \mathbb{R}$.

Theorem 1.1. *For any $p \in (0,1)$ and $n \in \mathbb{N}$, the added value $A_p(\boldsymbol{p}_n)$ exists as an improper Riemann integral if and only if $\boldsymbol{p}_n$ is admissible, in which case $A_p(\boldsymbol{p}_n) = \frac{p}{q} H_r(\boldsymbol{p}_n)$.*

Csörgő and Simons [17] proved this theorem for the classical St. Petersburg(1/2) game, played with an unbiased coin. However, in that case they proved the following stronger result: the independent St. Petersburg(1/2) variables $X_1, \dots, X_n$ can be defined on a rich enough probability space that carries, for each admissible strategy $\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})$, a St. Petersburg(1/2) random variable $X_{\boldsymbol{p}_n}$ and a nonnegative random variable $Y_{\boldsymbol{p}_n}$ such that $T_{\boldsymbol{p}_n} = p_{1,n}X_1 + \cdots + p_{n,n}X_n = X_{\boldsymbol{p}_n} + Y_{\boldsymbol{p}_n}$ almost surely. This implies the stochastic inequality $T_{\boldsymbol{p}_n} \geq_{\mathcal{D}} X_1$. Hence the integrand in $A_{1/2}(\boldsymbol{p}_n)$ is nonnegative and thus $A_{1/2}(\boldsymbol{p}_n)$ is trivially finite as a Lebesgue integral. As the next result shows, stochastic dominance is preserved for two players for an arbitrary St. Petersburg parameter $p \in (0,1)$.

Theorem 1.2. *For any $p \in (0,1)$, if $\boldsymbol{p}_2 = (q^a, q^b)$ is an admissible pooling strategy for some $a, b \in \mathbb{N}$, then $T_{\boldsymbol{p}_2} = q^a X_1 + q^b X_2$ is stochastically larger than X_1.*

Surprisingly, however, for $n \geq 3$ gamblers stochastic dominance generally fails to hold for admissible strategies. Our example to demonstrate this is when $p = (n-1)/n$, $q = 1-p = 1/n$, so that $r = 1/q = n$ is also the number of Pauls. Then $\mathbf{P}\{X = n^k\} = (n-1)/n^k$, $k \in \mathbb{N}$, and the averaging pooling strategy $\boldsymbol{p}_n = \boldsymbol{p}_n^\diamond = (1/n, 1/n, \dots, 1/n)$ is admissible. For this strategy the weighted sum is $T_{\boldsymbol{p}_n^\diamond} = (X_1 + \cdots + X_n)/n$, so that for $n = 2$ Theorem 1.2 says in particular that $S_2 = 2T_{\boldsymbol{p}_2^\diamond} = X_1 + X_2$ is stochastically larger than $2X_1$. This is not true for $n \geq 3$.

Theorem 1.3. *If $p = (n-1)/n$, $q = 1/n$ and $n \geq 3$, then neither $S_n = X_1 + \cdots + X_n$ nor nX_1 is stochastically larger than the other.*

In view of Theorem 1.2 the integrand in (1.1) is nonnegative whenever $\boldsymbol{p}_2$ is admissible, so that the integral $A_p(\boldsymbol{p}_2)$ described in Theorem 1.1 strengthens to that of a Lebesgue integral when $n = 2$. While the same conclusion holds for $n \geq 3$, Theorem 1.3 rules out so simple a line of reasoning.

Theorem 1.4. *For every parameter* $p \in (0, 1)$ *and every admissible strategy* $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ *the integral* $A_p(\boldsymbol{p}_n)$ *in* (1.1) *is finite as a Lebesgue integral.*

Theorem 1.1 characterizes the pooling strategies that yield added values. However, admissible strategies do *not* exist for all, in fact, for most parameters $p \in (0, 1)$. Call a parameter p *admissible*, if for p there exists an admissible strategy which is not individualistic. Theorem 1.1 then says that p is admissible if and only if for $q = 1 - p$ there exist positive integers $a_1 \geq a_2 \geq \cdots \geq a_k$, for some $k \in \mathbb{N}$, such that $q^{a_1} + q^{a_2} + \cdots + q^{a_k} = 1$. In this case, $r = 1/q$ is an algebraic integer. If $a_1 > a_2$, then q is also an algebraic integer, thus q is an algebraic unit. The set of algebraic numbers is countable, so there are at most a countable number of admissible parameters p. When $q = 1 - p$ is rational for an admissible $p \in (0, 1)$, the equation implies $q = 1/m$ for some integer $m \geq 2$. Thus the set of rational admissible parameters is $\{(m - 1)/m : m \geq 2\}$. In particular, it is interesting that the classical $p = 1/2$ is the smallest such St. Petersburg parameter. It follows that the set of all admissible parameters p is countable. Nevertheless, it can be shown that this set is dense in the interval $(0, 1)$:

Theorem 1.5. *The set of admissible parameters is dense in* $(0, 1)$.

When a given number of our Pauls happen to have admissible strategies, a natural question is: which is the best? In the latter rational case when $p = (m-1)/m$ for some integer $m \geq 2$, and so $r = 1/q = m \geq 2$ is an integer, the answer is given by the next result, in which $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ is the integer part, $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$ is the integer ceiling and $\langle x \rangle = x - \lfloor x \rfloor = x + \lceil -x \rceil$ is the fractional part of a number $x \in \mathbb{R}$.

Theorem 1.6. *If* $p = (r-1)/r$ *and* $n = r^{\lfloor \log_r n \rfloor} + (r-1)r_n$ *for some integers* $r \geq 2$ *and* $0 \leq r_n \leq r^{\lfloor \log_r n \rfloor} - 1$, *then*

$$A_p(\boldsymbol{p}_n) = \frac{p}{q}\, H_r(\boldsymbol{p}_n) \leq \frac{p}{q}\, \log_r n - \delta_p(n) =: A^*_{p,n} \tag{1.2}$$

for every admissible strategy $\boldsymbol{p}_n$, *where* $\delta_p(u) = 1 + (r - 1)\langle \log_r u \rangle - r^{\langle \log_r u \rangle}$, $u > 0$. *Moreover, the bound* $A^*_{p,n}$ *is attainable by means of the admissible strategy*

$$\boldsymbol{p}^*_n = (p^*_{1,n}, \ldots, p^*_{n,n}) = (rp^*_n, \ldots, rp^*_n, p^*_n, \ldots, p^*_n) \quad \textit{with} \quad p^*_n = \frac{1}{r^{\lceil \log_r n \rceil}},$$

where the number of p_n^* *s and* rp_n^* *s are, respectively,*

$$m_{1,p}(n) = \frac{rn - r^{\lceil \log_r n \rceil}}{r-1} \quad \textit{and} \quad m_{2,p}(n) = \frac{r^{\lceil \log_r n \rceil} - n}{r-1}.$$

Apart from reorderings of the components of $\boldsymbol{p}_n^*$*, the point of maximum is unique.*

The continuous function $\delta_p(\cdot)$ is nonnegative, its maximum is given in formula (3.4) of Csörgő and Simons [14].

It is easy to see that if n is not in the form $r^{\lfloor \log_r n \rfloor} + (r-1)r_n$, then 0 must be included among the components of the strategy, which does not increase the entropy. So it is enough to investigate the number of players in the form above. Here are the first few optimal strategy and the corresponding added value, when $r = 3$:

$$\begin{aligned}
A_3(\boldsymbol{p}_3^*) &= 2, & \boldsymbol{p}_3^* &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\
A_3(\boldsymbol{p}_4^*) &= 2, & \boldsymbol{p}_4^* &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right) \\
A_3(\boldsymbol{p}_5^*) &= 2\tfrac{2}{3}, & \boldsymbol{p}_5^* &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right) \\
A_3(\boldsymbol{p}_6^*) &= 2\tfrac{2}{3}, & \boldsymbol{p}_6^* &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0\right) \\
A_3(\boldsymbol{p}_7^*) &= 3\tfrac{1}{3}, & \boldsymbol{p}_7^* &= \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right) \\
A_3(\boldsymbol{p}_8^*) &= 3\tfrac{1}{3}, & \boldsymbol{p}_8^* &= \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0\right) \\
A_3(\boldsymbol{p}_9^*) &= 4, & \boldsymbol{p}_9^* &= \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right) \\
A_3(\boldsymbol{p}_{10}^*) &= 4, & \boldsymbol{p}_{10}^* &= \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0\right) \\
A_3(\boldsymbol{p}_{11}^*) &= 4\tfrac{2}{9}, & \boldsymbol{p}_{11}^* &= \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}\right) \\
A_3(\boldsymbol{p}_{12}^*) &= 4\tfrac{2}{9}, & \boldsymbol{p}_{12}^* &= \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, 0\right) \\
A_3(\boldsymbol{p}_{13}^*) &= 4\tfrac{4}{9}, & \boldsymbol{p}_{13}^* &= \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}\right).
\end{aligned}$$

Theorem 1.6 is not applicable for an irrational p. On the other hand, in every admissible situation $A_p(\boldsymbol{p}_n) = (r-1)H_r(\boldsymbol{p}_n)$ by Theorem 1.3, and the trivial upper bound $H_r(\boldsymbol{p}_n) \leq \log_r n = H_r(\boldsymbol{p}_n^\diamond)$ is valid for the entropy of every $\boldsymbol{p}_n$, where $\boldsymbol{p}_n^\diamond = (1/n, 1/n, \ldots, 1/n)$. However equality cannot hold in general because $\boldsymbol{p}_n^\diamond$ is not admissible for every admissible parameter p. Apart from those cases which can be reduced to the rational case, that is when $q = 1 - p = 1/\sqrt[k]{m}$ for some integers $m, k \geq 2$, the problem of the best admissible strategy is unsolved.

For the irrational case the simplest example is the equation $q^2 + q = 1$, the solution of which is $q = \tau := (\sqrt{5} - 1)/2 \approx 0.618$, the ratio of golden section. Thus, pertaining to the irrational parameter $p^\star = (3 - \sqrt{5})/2 \approx 0.382$, the vector (τ^2, τ) is an admissible strategy for two players. From this strategy we can generate admissible strategies for an arbitrary number of players. Indeed, substituting $\tau^3 + \tau^2 = \tau$ for τ, and $\tau^4 + \tau^3 = \tau^2$ for τ^2, we obtain (τ^3, τ^2, τ^2) and (τ^4, τ^3, τ), both admissible strategies for three Pauls. Continuing this algorithm, each time substituting $\tau^{m+2} + \tau^{m+1}$ for τ^m if the exponent m is present, after l steps we obtain admissible strategies for $2+l$ gamblers, $l \in \mathbb{N}$. However, even if we allow all possible branches generated by this algorithm, the result is incomplete in the sense that there are admissible strategies, such as $(\tau^8, \tau^5, \tau^5, \tau^5, \tau^3, \tau^3, \tau^3)$ for seven Pauls, that are avoided. Consider all the strategies that can be generated by the branching algorithm from (τ^2, τ), and for every $n \geq 2$ call the best among these *conditionally optimal*, denoted by $\boldsymbol{p}_n^\star$. Let f_n be the n^{th} Fibonacci number, so that with $f_0 = 1$, $f_1 = 1$ and $f_{n+1} = f_{n-1} + f_n$, $n \in \mathbb{N}$. We can show that the conditionally optimal strategy for $f_n + k$ players, $k \in \{0, 1, \ldots, f_{n-1} - 1\}$, each playing a St. Petersburg$(p^\star)$ game, is

$$\boldsymbol{p}_{f_n+k}^\star = \big(\underbrace{\tau^{n+1}, \ldots, \tau^{n+1}}_{k \text{ times}}, \underbrace{\tau^{n}, \ldots, \tau^{n}}_{f_{n-2}+k \text{ times}}, \underbrace{\tau^{n-1}, \ldots, \tau^{n-1}}_{f_{n-1}-k \text{ times}}\big),$$

with the corresponding added value $A_p(\boldsymbol{p}_{f_n+k}^\star) = \tau^n[k(2-\tau) + nf_{n-2}\tau + (n-1)f_{n-1}]$. Because of the inherent number-theoretic difficulties, we do not know whether these conditionally optimal strategies are optimal in general. Here are the first few conditionally optimal strategies and the corresponding added values:

$$\begin{aligned}
A_p(\boldsymbol{p}_2^*) &= 3\tau - 1 \approx 0.854, & \boldsymbol{p}_2^* &= \big(\tau^2, \tau\big) \\
A_p(\boldsymbol{p}_3^*) &= 2 - \tau \approx 1.382, & \boldsymbol{p}_3^* &= \big(\tau^3, \tau^2, \tau^2\big) \\
A_p(\boldsymbol{p}_4^*) &= 6\tau - 2 \approx 1.708, & \boldsymbol{p}_4^* &= \big(\tau^4, \tau^3, \tau^3, \tau^2\big) \\
A_p(\boldsymbol{p}_5^*) &= 13\tau - 6 \approx 2.034, & \boldsymbol{p}_5^* &= \big(\tau^4, \tau^4, \tau^3, \tau^3, \tau^3\big) \\
A_p(\boldsymbol{p}_6^*) &= 1 + 2\tau \approx 2.236, & \boldsymbol{p}_6^* &= \big(\tau^5, \tau^4, \tau^4, \tau^4, \tau^3, \tau^3\big).
\end{aligned}$$

Finally, we show that an extended form of our branching algorithm has an interesting property concerning stochastic domination. For any admissible parameter $p \in (0,1)$, let $(q^{a_1}, q^{a_2}, \dots, q^{a_n})$ and $(q^{b_1}, q^{b_2}, \dots, q^{b_m})$ be admissible strategies for n and m Pauls for any $n, m \geq 2$. Substituting $q^{a_k+b_1} + q^{a_k+b_2} + \cdots + q^{a_k+b_m} = q^{a_k}$ for q^{a_k}, where $k \in \{1, \dots, n\}$ is arbitrary, we obtain a strategy $(q^{d_1}, q^{d_2}, \dots, q^{d_{n+m-1}})$ for $n+m-1$ gamblers, where the sequence $d_1 \geq d_2 \geq \cdots \geq d_{n+m-1}$ is a nonincreasing rearrangement of the sequence $a_1, \dots, a_{k-1}, a_k + b_1, \dots, a_k + b_m, a_{k+1}, \dots, a_n$. We say that a strategy $\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})$ is *stochastically dominant* if $p_{1,n}X_1 + \cdots + p_{n,n}X_n \geq_{\mathcal{D}} X_1$. The last theorem states that the branching algorithm preserves stochastic dominance. Choosing first $n = m = 2$, it may be used in conjunction with Theorem 1.2 as a starting point.

Theorem 1.7. *If the strategies $(q^{a_1}, q^{a_2}, \dots, q^{a_n})$ and $(q^{b_1}, q^{b_2}, \dots, q^{b_m})$ are both stochastically dominant, then the generated strategy $(q^{d_1}, q^{d_2}, \dots, q^{d_{n+m-1}})$ is also stochastically dominant.*

All our results here are for fixed numbers of players. Csörgő and Simons [16] proved for an arbitrary sequence of strategies $\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})$ that $(p_{1,n}X_1 + \cdots + p_{n,n}X_n)/H_r(\boldsymbol{p}_n)$ converges in probability to p/q, as $n \to \infty$, whenever $H_r(\boldsymbol{p}_n) \to \infty$.

1.4. Proofs

The first four lemmas are needed for the proof of Theorem 1.1, while the fifth lemma is used in the proof of Theorem 1.6. The first two lemmas are the same as Lemmas 1 and 2 in Csörgő and Simons [17], therefore we use it without proof, while the others are the generalizations of Lemmas 3, 4 and 5 in [17].

Lemma 1.1. *Let U and V be nonnegative random variables, and assume that $\mathbf{E}(\min(U,V))$ is finite. Then*

$$\int_0^\infty \big[\mathbf{P}\{U+V>x\} - \mathbf{P}\{U>x\} - \mathbf{P}\{V>x\}\big]\mathrm{d}x = 0,$$

where the integrals is Lebesgue-integral.

Lemma 1.2. *Let $U_1, \dots, U_n$ be nonnegative random variables, and assume that $\mathbf{E}\big(\min(U_i, U_j)\big) < \infty$, for all $i, j = 1, \dots, n$, $i \neq j$. Then*

$$\int_0^\infty \left[\mathbf{P}\left\{\sum_{j=1}^n U_j > x\right\} - \sum_{j=1}^n \mathbf{P}\{U_j > x\}\right]\mathrm{d}x = 0,$$

where the integral is Lebesgue-integral.

Lemma 1.3. *If X_1, X_2 are independent St. Petersburg(p) random variables and c_1 and c_2 positive constants, then* $\mathbf{E}(\min(c_1X_1, c_2X_2)) < \infty$.

Proof. We know from [14] and [16] that $1-F_p(x) = \mathbf{P}\{X > x\} = q^{\lfloor \log_r x \rfloor} = r^{\langle \log_r x \rangle}/x$ for all $x \geq r$, and 1 otherwise. Hence, if $x \geq r\max(c_1, c_2)$, then the inequality

$$\mathbf{P}\{\min(c_1X_1, c_2X_2) > x\} = \mathbf{P}\{c_1X_1 > x\}\,\mathbf{P}\{c_2X_2 > x\} < c_1c_2r^2/x^2$$

holds and, therefore,

$$\mathbf{E}(\min(c_1X_1, c_2X_2)) = \int_0^\infty \mathbf{P}\{\min(c_1X_1, c_2X_2) > x\}\,\mathrm{d}x < \infty\,.$$

■

Lemma 1.4. *If X is a St. Petersburg(p) random variable and $b \geq 1$, then*

$$\int_0^b P\{X > x\}\,\mathrm{d}x = (r-1)\lfloor \log_r b \rfloor + r^{\langle \log_r b \rangle} = 1 + (r-1)\log_r b - \delta_p(b),$$

where the function $\delta_p(\cdot)$ is defined in Theorem 1.6.

Proof. Notice that $1 = 1-F_p(x) = \mathbf{P}\{X > x\} = q^{\lfloor \log_r x \rfloor}$ even for $x \in [1, r)$. So what we need to prove is that $\int_1^b q^{\lfloor \log_r x \rfloor}\,\mathrm{d}x = (r-1)\lfloor \log_r b \rfloor + r^{\langle \log_r b \rangle} - 1$ for $b > 1$. If $c = \log_r b > 0$, then

$$\begin{aligned}
\int_1^{r^c} q^{\lfloor \log_r x \rfloor}\,\mathrm{d}x &= \int_0^c q^{\lfloor y \rfloor} r^y \log r\,\mathrm{d}y = (\log r)\int_0^c r^{\langle y \rangle}\mathrm{d}y \\
&= (\log r)\left[\lfloor c \rfloor \int_0^1 r^y\mathrm{d}y + \int_{\lfloor c \rfloor}^c r^{\langle y \rangle}\mathrm{d}y\right] \\
&= \lfloor c \rfloor(r-1) + (\log r)\int_0^{\langle c \rangle} r^y\,\mathrm{d}y = (r-1)\lfloor c \rfloor + r^{\langle c \rangle} - 1,
\end{aligned}$$

where $\log = \log_\mathrm{e}$ is the natural logarithm, which is the desired equation. ■

Lemma 1.5. *If $r \in \{2, 3, \ldots\}$, then the number of the smallest strictly positive components of an admissible strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ is divisible by r.*

Proof. Let the smallest strictly positive component be $1/r^k$ for some $k \in \mathbb{N}$. Since $\sum_{j=1}^{n} p_{j,n} r^k = r^k$, the sum must be divisible by r, so the number of terms equal to 1 in the sum, which is the number of the components $1/r^k$ in $\boldsymbol{p}_n$, is also divisible by r. ■

Proof of Theorem 1.1. With the extended Lemmas 1.3 and 1.4, the proof is an easy generalization of that in the classical case $p = 1/2$ in [17].

For a given strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$, the integral $A_p(\boldsymbol{p}_n)$ in (1.1) is defined in the improper Riemann sense if and only if $A_p(\boldsymbol{p}_n, b) \to A_p(\boldsymbol{p}_n)$ as $b \to \infty$, where

$$A_p(\boldsymbol{p}_n, b) = \int_0^b \big[\mathbf{P}\{p_{1,n}X_1 + \cdots + p_{n,n}X_n > x\} - \mathbf{P}\{X_1 > x\}\big]\,\mathrm{d}x.$$

Using Lemma 1.2 and 1.3, we have

$$\int_0^\infty \left[\mathbf{P}\{p_{1,n}X_1 + \cdots + p_{n,n}X_n > x\} - \sum_{j=1}^{n} \mathbf{P}\{p_{j,n}X_j > x\}\right]\mathrm{d}x = 0,$$

thus

$$\int_0^b \mathbf{P}\left\{\sum_{j=1}^{n} p_{j,n}X_j > x\right\}\mathrm{d}x = \sum_{j=1}^{n} \int_0^b \mathbf{P}\{p_{j,n}X_j > x\}\,\mathrm{d}x + o(1),$$

where $o(1) \to 0$, as $b \to \infty$. Operating in the sum with those terms, in which

$p_{j,n} > 0$, using Lemma 1.4 we obtain

$$\begin{aligned}
A_p(\boldsymbol{p}_n, b) &= \int_0^b \mathbf{P}\bigg\{\sum_{j=1}^n p_{j,n}X_j > x\bigg\}\mathrm{d}x - \int_0^b \mathbf{P}\{X > x\}\,\mathrm{d}x \\
&= \sum_{j=1}^n \int_0^b \mathbf{P}\{p_{j,n}X_j > x\}\,\mathrm{d}x - \int_0^b \mathbf{P}\{X > x\}\,\mathrm{d}x + o(1) \\
&= \sum_{j=1}^n p_{j,n}\int_0^{\frac{b}{p_{j,n}}} \mathbf{P}\{X > y\}\,\mathrm{d}y - \int_0^b \mathbf{P}\{X > x\}\,\mathrm{d}x + o(1) \\
&= \sum_{j=1}^n p_{j,n}\bigg\{\bigg\lfloor \log_r \frac{b}{p_{j,n}}\bigg\rfloor(r-1) + r^{\left\langle \log_r \frac{b}{p_{j,n}}\right\rangle}\bigg\} \\
&\quad - \lfloor \log_r b\rfloor(r-1) - r^{\langle \log_r b\rangle} + o(1) \\
&= (r-1)\sum_{j=1}^n p_{j,n}\bigg\{\log_r \frac{b}{p_{j,n}} - \bigg\langle \log_r \frac{b}{p_{j,n}}\bigg\rangle - \lfloor \log_r b\rfloor\bigg\} \\
&\quad + \sum_{j=1}^n p_{j,n} r^{\left\langle \log_r \frac{b}{p_{j,n}}\right\rangle} - r^{\langle \log_r b\rangle} + o(1) \\
&= (r-1)H_r(\boldsymbol{p}_n) - \sum_{j=1}^n p_{j,n}\bigg\{1 - (r-1)\bigg\langle \log_r \frac{b}{p_{j,n}}\bigg\rangle + r^{\left\langle \log_r \frac{b}{p_{j,n}}\right\rangle}\bigg\} \\
&\quad + 1 - (r-1)\langle \log_r b\rangle + r^{\langle \log_r b\rangle} + o(1),
\end{aligned}$$

that is $A_p(\boldsymbol{p}_n, b) = (r-1)H_r(\boldsymbol{p}_n) + R_r(\boldsymbol{p}_n, b) + o(1)$ as $b \to \infty$, where $R_r(\boldsymbol{p}_n, b) = \delta_p(b) - \sum_{j=1}^n p_{j,n}\,\delta_p(b/p_{j,n})$, and the function δ_p is defined in Theorem 1.6. Notice that $\delta_p(ur^k) = \delta_p(u)$, $u > 0$, for every $k \in \mathbb{Z}$. Thus if $\boldsymbol{p}_n$ is admissible, then

$$R_r(\boldsymbol{p}_n, b) = \delta_p(b) - \sum_{j=1}^n p_{j,n}\,\delta_p\bigg(\frac{b}{p_{j,n}}\bigg) = \delta_p(b) - \sum_{j=1}^n p_{j,n}\,\delta_p(b) = 0,$$

and hence $A_p(\boldsymbol{p}_n, b) = (r-1)H_r(\boldsymbol{p}_n) + o(1)$ as $b \to \infty$, which is the "if part" of the theorem.

Conversely, suppose that $A_p(\boldsymbol{p}_n)$ in (1.1) exists, so that $A_p(\boldsymbol{p}_n, b) \to A_p(\boldsymbol{p}_n)$ as $b \to \infty$. Using the above periodicity property of $\delta_p(\cdot)$, we get $R_r(\boldsymbol{p}_n, r^k b) = R_r(\boldsymbol{p}_n, b)$ for every $k \in \mathbb{Z}$. Fixing $b > 0$ and letting $k \to \infty$, so that $r^k b \to \infty$, we get $R_r(\boldsymbol{p}_n, b) = A_p(\boldsymbol{p}_n) - (r-1)H_r(\boldsymbol{p}_n)$. Let $D = D_+ - D_-$, where D_+ and D_- are the right-side and left-side differential operators, respectively. Then one can compute easily that

$$D\delta_p(s) = \begin{cases} \frac{r-1}{r^k}, & \text{for } s = r^k \text{ when } k \in \mathbb{Z}, \\ 0, & \text{for all other } s > 0, \end{cases}$$

from which, for all $j \in \{1, \dots, n\}$ for which $p_{j,n} > 0$, we find that

$$D\, p_{j,n}\, \delta_p\!\left(\frac{b}{p_{j,n}}\right) = \begin{cases} \frac{r-1}{r^k}, & \text{for } b = r^k\, p_{j,n} \text{ when } k \in \mathbb{Z}, \\ 0, & \text{for all other } b > 0. \end{cases}$$

Consequently, we have

$$DR_r(\boldsymbol{p}_n, b)\big|_{b=1} = r - 1 - (r-1) \sum_{j \in A} p_{j,n},$$

where A is the set of indices $j \in \{1, \dots, n\}$ for which $p_{j,n}$ is an integer power of r. Since, on the other hand, $DR_r(\boldsymbol{p}_n, b) = 0$, this implies $\sum_{j \in A} p_{j,n} = 1$, and thus completes the proof. ∎

Proof of Theorem 1.2. Let us assume that $q^a + q^b = 1$ for some $a, b \in \mathbb{N}$. Then $\mathbf{P}\{X_1 \le r^k\} = F_p(r^k) = 1 - q^k$ for every $k \in \mathbb{N}$. We estimate the probability $\mathbf{P}\{T_2 \le r^k\}$, where $T_2 = q^a X_1 + q^b X_2$. If $T_2 \le r^k$, then

(1) $X_1, X_2 \le r^k$, or
(2) $X_1 = r^{k+1}, \dots, r^{k+a-1}$ and $X_2 \le r^{k-1}$, or
(3) $X_2 = r^{k+1}, \dots, r^{k+b-1}$ and $X_1 \le r^{k-1}$.

We obtain

$$\begin{aligned} \mathbf{P}\big\{T_2 \le r^k\big\} &\le \big(1 - q^k\big)^2 + \big(1 - q^{k-1}\big)q^k\big(1 - q^{a-1}\big) + \big(1 - q^{k-1}\big)q^k\big(1 - q^{b-1}\big) \\ &= \big(1 - q^k\big)^2 + \big(1 - q^{k-1}\big)q^k\left(2 - \frac{1}{q}\right) = 1 - q^{k-1} + q^{2k}\left(\frac{1}{q} - 1\right)^2. \end{aligned}$$

Since the distribution function of X_1 jumps only in the points $x = r^k$, it is enough to show that $\mathbf{P}\big\{T_2 < r^k\big\} \le \mathbf{P}\{X_1 < r^k\} = \mathbf{P}\{X_1 \le r^{k-1}\} = 1 - q^{k-1}$. This is true, because

$$\begin{aligned} \mathbf{P}\big\{T_2 < r^k\big\} &= \mathbf{P}\big\{T_2 \le r^k\big\} - \mathbf{P}\big\{T_2 = r^k\big\} \\ &\le 1 - q^{k-1} + q^{2k}\left(\frac{1}{q} - 1\right)^2 - \mathbf{P}\big\{X_1 = r^k,\ X_2 = r^k\big\} \\ &= 1 - q^{k-1}, \end{aligned}$$

completing the proof. ■

Proof of Theorem 1.3. We prove that stronger statement that the graphs of the distribution functions of S_n and nX_1 cross each other infinitely often. To be more precise we show that both $\mathbf{P}\{nX_1 \le n^k\} > \mathbf{P}\{S_n \le n^k\}$ and $\mathbf{P}\{nX_1 < n^k\} < \mathbf{P}\{S_n < n^k\}$ hold whenever $k \ge 3$.

Notice that the inequality $S_n \le n^k$ holds if and only if all the inequalities $X_1 \le n^{k-1}$, $X_2 \le n^{k-1}, \dots, X_n \le n^{k-1}$ hold. This implies for arbitrary $k \ge 2$ that

$$\begin{aligned}\mathbf{P}\big\{S_n \le n^k\big\} &= \mathbf{P}\bigg\{\bigcap_{j=1}^{n}\{X_j \le n^{k-1}\}\bigg\} = \Big(\frac{n-1}{n} + \frac{n-1}{n^2} + \dots + \frac{n-1}{n^{k-1}}\Big)^n \\ &= \Big(1 - \frac{1}{n^{k-1}}\Big)^n.\end{aligned}$$

Clearly, $\mathbf{P}\big\{nX_1 \le n^k\big\} = \mathbf{P}\big\{X_1 \le n^{k-1}\big\} = 1 - 1/n^{k-1}$, so $\mathbf{P}\big\{nX_1 \le n^k\big\} > \mathbf{P}\big\{S_n \le n^k\big\}$.

Now consider the probabilities $\mathbf{P}\{nX_1 < n^k\}$ and $\mathbf{P}\{S_n < n^k\}$. When $k = 2$, both of them are zero. So, assume $k \ge 3$. Noticing the equalities $\mathbf{P}\{S_n < n^k\} = \mathbf{P}\{S_n \le n^k\} - \mathbf{P}\{S_n = n^k\}$,

$$\mathbf{P}\big\{S_n = n^k\big\} = \mathbf{P}\big\{X_1 = n^{k-1}, X_2 = n^{k-1}, \dots, X_n = n^{k-1}\big\} = \Big(\frac{n-1}{n^{k-1}}\Big)^n,$$

and $\mathbf{P}\{nX_1 < n^k\} = \mathbf{P}\{nX_1 \le n^{k-1}\} = \mathbf{P}\{X_1 \le n^{k-2}\}$, we have

$$\mathbf{P}\big\{S_n < n^k\big\} = \Big(1 - \frac{1}{n^{k-1}}\Big)^n - \Big(\frac{n-1}{n^{k-1}}\Big)^n > 1 - \frac{1}{n^{k-2}} = \mathbf{P}\{nX_1 < n^k\},$$

where elementary calculation shows that the inequality holds for $n \ge 3$ and $k \ge 3$. ■

Proof of Theorem 1.4. Let $n \ge 2$ be the number of Pauls. By Theorem 1.1, for every admissible strategy $q = 1 - p$ satisfies the equation $q^{a_1} + q^{a_2} + \dots + q^{a_m} = 1$, where $a_1, a_2, \dots, a_m \in \mathbb{N}$ and $m \in \{2, 3, \dots, n\}$. Without loss of generality we assume that the zeros, if any, are the last components of the strategy, so that $\boldsymbol{p}_n = (q^{a_1}, q^{a_2}, \dots, q^{a_m}, 0, \dots, 0)$. Then $T_m := \sum_{j=1}^{n} p_{j,n} X_j = q^{a_1} X_1 + \dots + q^{a_m} X_m$. We estimate the probability $\mathbf{P}\{T_m \le r^k\}$. If the event $\{T_m \le r^k\}$ occurs, then we must have all the inequalities $X_1 \le r^{k+a_1-1}, X_2 \le r^{k+a_2-1}, \dots, X_m \le r^{k+a_m-1}$. Hence,

$$\begin{aligned}\mathbf{P}\big\{T_m \le r^k\big\} &\le \big(1 - q^{k+a_1-1}\big)\big(1 - q^{k+a_2-1}\big) \cdots \big(1 - q^{k+a_m-1}\big) \\ &= 1 - q^k\big(q^{a_1-1} + q^{a_2-1} + \dots + q^{a_m-1}\big) + q^{2k} C_2 + \dots + q^{mk} C_m \\ &= 1 - q^{k-1} + q^{2k} C_2 + \dots + q^{mk} C_m\,,\end{aligned}$$

where the constants $C_2, C_3, \ldots, C_m$ do not depend on k.

Since $\boldsymbol{p}_n$ is admissible, the integral $\int_0^\infty \big[\mathbf{P}\{T_m > x\} - \mathbf{P}\{X_1 > x\}\big]\,\mathrm{d}x$ exists as an improper Riemann integral. Hence it suffices to show that the integral of the negative part $g_m^-(x)$ of the function $g_m(x) := \mathbf{P}\{T_m > x\} - \mathbf{P}\{X_1 > x\}$ is finite. Notice that

$$g_m(x) = \mathbf{P}\{X_1 \le x\} - \mathbf{P}\{T_m \le x\} \ge \mathbf{P}\{X_1 < x\} - \mathbf{P}\{T_m \le x\} =: h_m(x)$$

for all $x > 0$. Clearly, the function $h_m(x)$ takes a minimum value on the interval $(r^{k-1}, r^k]$ at $x = r^k$, for which the estimate above yields

$$\begin{aligned} h_m(r^k) &= \mathbf{P}\big\{X_1 < r^k\big\} - \mathbf{P}\big\{T_m \le r^k\big\} \\ &\ge 1 - q^{k-1} - \big(1 - q^{k-1} + q^{2k}C_2 + \cdots + q^{mk}C_m\big) \\ &= -\big(q^{2k}C_2 + \cdots + q^{mk}C_m\big). \end{aligned}$$

Therefore, setting $C_1 = \int_0^1 h_m^-(x)\,\mathrm{d}x$, we obtain

$$\begin{aligned} \int_0^\infty g_m^-(x)\,\mathrm{d}x \le \int_0^\infty h_m^-(x)\mathrm{d}x &\le C_1 + \sum_{k=1}^\infty \int_{r^{k-1}}^{r^k} \big(q^{2k}|C_2| + \cdots + q^{mk}|C_m|\big)\mathrm{d}x \\ &= C_1 + \sum_{k=1}^\infty r^k \left(1 - \frac{1}{r}\right)\big(q^{2k}|C_2| + \cdots + q^{mk}|C_m|\big) \\ &= C_1 + (1-q)\sum_{k=1}^\infty \big(q^k|C_2| + \cdots + q^{(m-1)k}|C_m|\big) < \infty, \end{aligned}$$

which proves the theorem. ■

Proof of Theorem 1.5. We will prove the equivalent statement, that the set $\{q = 1 - p : p \text{ admissible}\}$ is dense. Let $(a, b) \subset (0, 1)$ be an arbitrary open interval. For the proof of the density, it is enough to construct a polynomial $P(x) = a_n x^n + \cdots + a_1 x - 1$, where $n \ge 1$ and $a_i \in \mathbb{N}$, $i = 1, \ldots, n$, such that $P(a) < 0 < P(b)$. Now, by monotonicity P has a root $q \in (a, b)$, so $p = 1 - q$ is admissible.

Let a_1 be the largest integer, such that $a_1 a < 1$. Then let a_2 be the largest integer, such that $a_1 a + a_2 a^2 < 1$. After defining $a_1, \ldots, a_{k-1}$ let a_k be the largest integer, such that $\sum_{j=1}^k a_j a^j < 1$. By the definition, $\sum_{j=1}^k a_j a^j \to 1$ as $k \to \infty$. So if n is large enough, we have

$$1 - a_1(b-a)/2 < \sum_{j=1}^n a_j a^j < \sum_{j=1}^n a_j b^j - a_1(b-a),$$

which is the desired inequality. ■

Proof of Theorem 1.6. This is based on the proof of Theorem 2 in [17], so we skip the details. Without loss of generality we assume that $\boldsymbol{p}_n$ is ordered: $p_{1,n} \geq p_{2,n} \geq \cdots \geq p_{n,n}$. The proof is by induction on r_n. For $r_n = 0$ the statement is true. Now suppose that all the statements of the theorem hold for $r_n - 1 \geq 0$, and consider the case $n = r^{\lfloor \log_r n \rfloor} + (r-1)r_n$. If $p_{n,n} = 0$, then we have at least $r-1$ zeros. Deleting them, we get a strategy $\hat{\boldsymbol{p}}_{n-(r-1)}$, and we are done in view of the fact that the bound $A^*_{p,n}$ in (1.2) is nondecreasing in n. In the other case, when $p_{n,n} = 1/r^k$ for some $k \in \mathbb{N}$, we have at least r of these smallest components by Lemma 1.5. Changing r of these to a single component $1/r^{k-1}$, we obtain a strategy $\hat{\boldsymbol{p}}_{n-(r-1)}$ for which $H_r(\boldsymbol{p}_n) - H_r(\hat{\boldsymbol{p}}_{n-(r-1)}) = 1/r^{k-1}$. Using the formula $A^*_{p,n} = (r-1)\lfloor \log_r n \rfloor + (r-1)r_n / r^{\lfloor \log_r n \rfloor}$, we have

$$A^*_{p,n} - A^*_{p,n-(r-1)} = \frac{r_n - r_{n-(r-1)}}{r^{\lfloor \log_r n \rfloor}},$$

and since $r_{n-(r-1)} = r_n - 1$, by the induction hypothesis we only have to show that

$$\frac{1}{r^{k-1}} \leq \frac{1}{r^{\lfloor \log_r n \rfloor}}.$$

Assume the contrary $1/r^{k-1} > 1/r^{\lfloor \log_r n \rfloor}$ which means $1/r^k \geq 1/r^{\lfloor \log_r n \rfloor}$. Thus we have

$$1 \leq \frac{r^{\lfloor \log_r n \rfloor}}{r^k} < \frac{n}{r^k}.$$

This is a contradiction, since $1/r^k$ was the smallest component, and the sum of the components is 1. To prove the uniqueness, notice that equality can hold only if the strategy $\hat{\boldsymbol{p}}_{n-(r-1)}$ is optimal and $1/r^{k-1} = 1/r^{\lfloor \log_r n \rfloor}$. By the induction hypothesis $\hat{\boldsymbol{p}}_{n-(r-1)}$ is uniquely determined, as the theorem states. Computing back the strategy $\boldsymbol{p}_n$ we obtain the desires form. ■

Lemma 1.6. *If U, V, W are independent random variables and $U \geq_{\mathcal{D}} V$, then $U + W \geq_{\mathcal{D}} V + W$.*

Proof. Let F, G and H be the distribution functions of U, V and W, respectively. By assumption, $F(x) \leq G(x)$ for all $x \in \mathbb{R}$. The random variables $U + W$ and $V + W$ have the distribution functions $F{*}H(\cdot)$ and $G{*}H(\cdot)$, where $*$ denotes Lebesgue–Stieltjes convolution. Thus

$$F{*}H(x) = \int_{-\infty}^{\infty} F(x-y)\mathrm{d}H(y) \leq \int_{-\infty}^{\infty} G(x-y)\mathrm{d}H(y) = G{*}H(x),$$

which proves the statement. ■

Proof of Theorem 1.7. Let $Y_1, \ldots, Y_m$, and $X_1, \ldots, X_n$ be independent St. Petersburg(p) variables. From the assumption we get

$$q^{a_k+b_1}Y_1 + q^{a_k+b_2}Y_2 + \cdots + q^{a_k+b_m}Y_m \ \geq_{\mathcal{D}} \ q^{a_k}X_k.$$

By Lemma 1.6 this implies

$$\begin{aligned} q^{a_1}X_1 \ + \cdots + q^{a_{k-1}}X_{k-1} + cq^{a_k+b_1}Y_1 + \cdots + q^{a_k+b_m}Y_m \\ +q^{a_{k+1}}X_{k+1} + \cdots + q^{a_n}X_n \\ \geq_{\mathcal{D}} \ q^{a_1}X_1 + \cdots + q^{a_{k-1}}X_{k-1} + q^{a_k}X_k + q^{a_{k+1}}X_{k+1} + \cdots + q^{a_n}X_n. \end{aligned}$$

Now the assumption and obvious transitivity together imply the theorem. ∎

Chapter 2.

Merging asymptotic expansions for generalized St. Petersburg games

2.1. Introduction

We further generalize the St. Petersburg game. Of course the roles of Peter and Pauls are the same: Peter offers to let Paul toss a possibly biased coin repeatedly until it lands heads and pays him $r^{k/\alpha}$ ducats if this happens on the k^{th} toss, $k \in \mathbb{N} = \{1, 2, \ldots\}$, where $r = 1/q$ for $q = 1-p$, and $p \in (0,1)$ is the probability of heads on each throw, while $\alpha > 0$ is a payoff parameter. (When $\alpha = 1$ we obtain the 'classical' generalized game, investigated in Chapter 1.) Thus if X denotes Paul's winning in this generalized St. Petersburg(α, p) game, then $\mathbf{P}\{X = r^{k/\alpha}\} = q^{k-1}p$, $k \in \mathbb{N}$.

Put $\lfloor y \rfloor = \max\{k \in \mathbb{Z} \colon k \le y\}$ and $\lceil y \rceil = \min\{k \in \mathbb{Z} \colon k \ge y\} = -\lfloor -y \rfloor$ for the usual integer part and 'ceiling' and $\langle y \rangle = y - \lfloor y \rfloor = y + \lceil -y \rceil$ for the fractional part of a number $y \in \mathbb{R}$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $\mathbb{R}$ is the real line. Then the generalized St. Petersburg distribution function of a single gain is

$$(2.1) \quad F_{\alpha,p}(x) = \mathbf{P}\{X \le x\} = \begin{cases} 0, & \text{if } x < r^{1/\alpha}, \\ 1 - q^{\lfloor \alpha \log_r x \rfloor} = 1 - \frac{r^{\langle \alpha \log_r x \rangle}}{x^\alpha}, & \text{if } x \ge r^{1/\alpha}, \end{cases}$$

where $\log_r$ stands for the logarithm to the base r, as before. For $\alpha = 1$ this is the same as F_p in Section 1.2.. We see that the payoff parameter $\alpha > 0$ is in fact a tail parameter of the distribution. In particular, $\mathbf{E}(X^\alpha) = \infty$, but $\mathbf{E}(X^\beta) = p/(q^{\beta/\alpha} - q)$ is finite for $\beta \in (0, \alpha)$, so for $\alpha > 2$ Paul's gain X has a finite variance, so Lévy's central limit theorem holds. As it was pointed out in [7] even for $\alpha = 2$ the St. Petersburg$(2, p)$ distribution is in the domain of attraction of the normal law. This can be checked by straightforward calculation, using the well-known characterization of the domain of attraction of the normal law ([23], or see the proof of Theorem 4.3). Hence for the problems to be entertained in this chapter the case $\alpha \geq 2$ is either not interesting or at least substantially different from the more difficult case $\alpha < 2$. Therefore, just as in [7] and [10] from now on we assume that $\alpha \in (0, 2)$. Of course, the most interesting case of this is when $\alpha \leq 1$, for which the mean is infinite.

In the followings we are interested in the asymptotic properties of the linear combinations. We already noted in Chapter 1 that there is no limit theorem for the sums. This holds also for the generalized version. Since the bounded oscillating function $r^{\langle \alpha \log_r x \rangle}$ in the numerator of (2.1) is not slowly varying at infinity, by the classical Doeblin – Gnedenko criterion the underlying generalized St. Petersburg distribution is *not* in the domain of attraction of any stable law. That is there is no asymptotic distribution for $(S_n - c_n)/a_n$, in the usual sense, whatever the centering and norming constants are. This is where the main difficulty lies for all generalized St. Petersburg games, when $\alpha < 2$.

However, asymptotic distributions do exist along subsequences of $\mathbb{N}$. In the classical case Martin-Löf [31] 'clarified the St. Petersburg paradox', showing that $S_{2^k}/2^k - k$ converge in distribution, as $k \to \infty$. It turned out in [11] that there are continuum different types of asymptotic distributions of $S_n/n - \log_2 n$ along different subsequences of $\mathbb{N}$. As Csörgő wrote [6] there are continuum many different clarification of the St. Petersburg paradox. The class of distribution functions of these possible limits may be given in the form $\{G_{1,1/2,\gamma}(\cdot) : \gamma \in (1/2, 1]\}$, where the values of the parameter γ enter as circular subsequential limits of $\gamma_n = n/2^{\lfloor \log_2 n \rfloor} \in (1/2, 1]$, which describes the location of n between two consecutive powers of 2. After these results it is tempting to pick up $G_{1,1/2,\gamma_n}(x)$ to approximate $\mathbf{P}\{S_n/n - \log_2 n \leq x\}$. The accurate version of this conjecture is the merging theorem $\sup_{x \in \mathbb{R}} |\mathbf{P}\{S_n/n - \log_2 n \leq x\} - G_{1,1/2,\gamma_n}(x)| \to 0$, which was showed by Csörgő in the general case in [7]. The optimality of the merge rates was proved by short asymptotic expansions by Csörgő in [10]. Later complete expansions were obtained by Pap [34]. Motivated by the latter results merging asymptotic expansions for semistable distributions were proven by the author [27]. The aim of this chapter is to generalize these results for linear combinations of independent St. Petersburg variables, that is to examine the gain of Paul_1, in the n-Paul problem.

2.2. Approximating semistable classes

Since we are interested in asymptotic results, we need to introduce some limiting quantities.

For the bias parameter $p \in (0,1)$, the payoff or tail parameter $\alpha \in (0,2)$ and a third parameter $\gamma \in (q,1]$, consider the infinitely divisible random variable

$$W_\gamma^{\alpha,p} = \frac{1}{\gamma^{1/\alpha}} \left\{ \sum_{m=0}^{-\infty} r^{m/\alpha} \left[Y_m^{p,\gamma} - \frac{p\gamma}{qr^m} \right] + \sum_{m=1}^{\infty} r^{m/\alpha}\, Y_m^{p,\gamma} \right\} + s_\gamma^{\alpha,p}, \tag{2.2}$$

where $\ldots, Y_{-2}^{p,\gamma}, Y_{-1}^{p,\gamma}, Y_0^{p,\gamma}, Y_1^{p,\gamma}, Y_2^{p,\gamma}, \ldots$ are independent random variables such that

$$\mathbf{P}\{Y_m^{p,\gamma} = k\} = \frac{(pr\gamma q^m)^k}{k!}\, \mathrm{e}^{-pr\gamma q^m}, \quad k = 0,1,2,\ldots,$$

that is, $Y_m^{p,\gamma}$ has the Poisson distribution with mean $pr\gamma q^m = p\gamma/(qr^m)$, $m \in \mathbb{Z}$, and where

$$s_\gamma^{\alpha,p} = \begin{cases} -\frac{p\gamma^{(\alpha-1)/\alpha}}{q^{1/\alpha}-q} = \frac{p}{q-q^{1/\alpha}}\, \frac{1}{\gamma^{(1-\alpha)/\alpha}}, & \text{if } \alpha \neq 1, \\ -\frac{p}{q} \log_r \gamma = \frac{p}{q} \log_r \frac{1}{\gamma}, & \text{if } \alpha = 1. \end{cases}$$

Kolmogorov's three series theorem implies that both infinite series in 2.2 converge almost surely. Let $G_{\alpha,p,\gamma}(x) = \mathbf{P}\{W_\gamma^{\alpha,p} \le x\}$, $x \in \mathbb{R}$, denote its distribution function. As derived in [7], pp. 821–823, its characteristic function is

$$\mathbf{g}_{\alpha,p,\gamma}(t) = \mathbf{E}\big(\mathrm{e}^{\mathrm{i}tW_\gamma^{\alpha,p}}\big) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, \mathrm{d}G_{\alpha,p,\gamma}(x) = \mathrm{e}^{y_\gamma^{\alpha,p}(t)}, \quad t \in \mathbb{R}, \tag{2.3}$$

where

$$y_\gamma^{\alpha,p}(t) = \mathrm{i}ts_\gamma^{\alpha,p} + \sum_{l=0}^{-\infty} \left(\exp\left\{ \frac{\mathrm{i}tr^{\frac{l}{\alpha}}}{\gamma^{\frac{1}{\alpha}}} \right\} - 1 - \frac{\mathrm{i}tr^{\frac{l}{\alpha}}}{\gamma^{\frac{1}{\alpha}}} \right) \frac{p\gamma}{qr^l} + \sum_{l=1}^{\infty} \left(\exp\left\{ \frac{\mathrm{i}tr^{\frac{l}{\alpha}}}{\gamma^1\alpha} \right\} - 1 \right) \frac{p\gamma}{qr^l}$$

$$= \exp\left\{ \mathrm{i}t\big[s_\gamma^{\alpha,p} + u_\gamma^{\alpha,p}\big] + \int_0^\infty \left(\mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2} \right) \mathrm{d}R_\gamma^{\alpha,p}(x) \right\} \tag{2.4}$$

with the finite constant

$$u_\gamma^{\alpha,p} = \frac{p\gamma^{(\alpha+1)/\alpha}}{q} \sum_{l=1}^{\infty} \frac{r^{(1-\alpha)l/\alpha}}{\gamma^{2/\alpha} + r^{2l/\alpha}} - \frac{p\gamma^{(\alpha-1)/\alpha}}{q} \sum_{l=0}^{\infty} \frac{1}{\gamma^{2/\alpha} r^{(3-\alpha)l/\alpha} + r^{(1-\alpha)l/\alpha}}$$

and right-hand-side Lévy function

$$R_\gamma^{\alpha,p}(x) = -\gamma q^{\lfloor \log_r(\gamma x^\alpha)\rfloor} = -\frac{\gamma}{r^{\lfloor \log_r(\gamma x^\alpha)\rfloor}} = -\frac{r^{\langle \log_r(\gamma x^\alpha)\rangle}}{x^\alpha}\,, \quad x > 0.$$

The integral form of the exponent of the characteristic function immediately implies that for every $p \in (0,1)$ and $\gamma \in (q,1]$ the infinitely divisible distribution of $W_\gamma^{\alpha,p}$ is semistable with exponent α; for the theory of semistable distributions required here we refer to [32], [13] and [9]; but there is a short introduction to the notion of semistability in Chapter 3. It follows that $G_{\alpha,p,\gamma}(\cdot)$ is infinitely many times differentiable and by classical results of Kruglov, recently exposed in [8], $\mathbf{E}(|W_\gamma^{\alpha,p}|^\alpha) = \infty$, but, for all $p \in (0,1)$ and $\gamma \in (q,1]$, the absolute moment

$$\text{(2.5)} \qquad \mathbf{E}\Big(|W_\gamma^{\alpha,p}|^\beta\Big) = \int_{-\infty}^{\infty} |x|^\beta \,\mathrm{d}G_{\alpha,p,\gamma}(x) = \int_{-\infty}^{\infty} |x|^\beta g_{\alpha,p,\gamma}(x)\,\mathrm{d}x < \infty$$

if $\beta \in (0,\alpha)$, with the density function $g_{\alpha,p,\gamma}(\cdot) = G'_{\alpha,p,\gamma}(\cdot) = G^{(2.1)}_{\alpha,p,\gamma}(\cdot)$.

As we noted before, the function $x \mapsto r^{\langle \alpha \log_r x\rangle}$ in (2.1) is not slowly varying at infinity, and hence it follows by the classical Doeblin – Gnedenko criterion that $F_{\alpha,p}(\cdot)$ in (2.1) is not in the domain of attraction of any (stable) distribution, that is, the cumulative winnings S_n cannot be centered and normalized to have a proper limiting distribution as $n \to \infty$ over the entire sequence $\mathbb{N}$ of natural numbers. However, it turned out in [31] and [11] that asymptotic distributions do exist along subsequences of $\mathbb{N}$ when $\alpha = 1$ and $p = 1/2$. In fact, subsequential limiting distributions exist for all $\alpha \in (0,2)$ and $p \in (0,1)$ for the sequence

$$\text{(2.6)} \quad F_n^{\alpha,p}(x) = \mathbf{P}\bigg\{\frac{S_n - c_n^{\alpha,p}}{n^{1/\alpha}} \le x\bigg\}, \text{ where } c_n^{\alpha,p} = \begin{cases} \frac{p\,n}{q^{1/\alpha}-q}\,, & \text{if } \alpha \neq 1, \\ \frac{p}{q}\, n \,\log_r n\,, & \text{if } \alpha = 1, \end{cases}$$

and are regulated by the position parameter

$$\text{(2.7)} \qquad \gamma_n = \frac{n}{r^{\lceil \log_r n\rceil}} \in (q,1],$$

which describes the location of $n = \gamma_n r^{\lceil \log_r n\rceil} \in \mathbb{N}$ between two consecutive powers of $r = 1/q$. As an extension of one of the results in [11] it can be shown that for any given subsequence $\{n_k\}_{k=1}^\infty$ of $\mathbb{N}$, the sequence $F_{n_k}^{\alpha,p}(\cdot)$ converges weakly as $k \to \infty$ if and only if $\gamma_{n_k} \xrightarrow{\text{cir}} \gamma$ for some $\gamma \in (q,1]$, where we write $\gamma_{n_k} \xrightarrow{\text{cir}} \gamma$ if $\lim_{k\to\infty} \gamma_{n_k} = \gamma$ for $\gamma \in (q,1]$, but we also write $\gamma_{n_k} \xrightarrow{\text{cir}} 1$ if either $\lim_{k\to\infty} \gamma_{n_k} = q$, or the sequence $\{\gamma_{n_k}\}_{k=1}^\infty$ has exactly two limit points, q and 1. If this circular convergence $\gamma_{n_k} \xrightarrow{\mathbf{cir}} \gamma$ takes place for some $\gamma \in (q,1]$, as $k \to \infty$, then $\lim_{k\to\infty} \sup_{x\in\mathbb{R}} |F_{n_k}^{\alpha,p}(x) - G_{\alpha,p,\gamma}(x)| = 0$.

The trouble with having many asymptotic distributions is resolved by the selection of a merging approximation to $F_n^{\alpha,p}(\cdot)$ for every $n \in \mathbb{N}$ from the class $\mathcal{G}^{\alpha,p} = \{G_{\alpha,p,\gamma}(\cdot) : q < \gamma \le 1\}$ of subsequential limits. The selection is given by the position parameter γ_n itself in (2.7), and we have the following merging theorem with rates [7]:

Theorem. *For every $\varepsilon > 0$ there exists a threshold $n_\varepsilon(\alpha,p) \in \mathbb{N}$, such that for $n > n_\varepsilon(\alpha,p)$*

$$\sup_{x\in\mathbb{R}} \left| \mathbf{P}\left\{ \frac{S_n - c_n^{\alpha,p}}{n^{1/\alpha}} \le x \right\} - G_{\alpha,p,\gamma_n}(x) \right| \le \begin{cases} (1+\varepsilon)\frac{C(\alpha,p)}{n}, & \text{if } 0 < \alpha < 1, \\ (1+\varepsilon)\frac{\pi}{8}\frac{p^2}{q^2}\frac{\lfloor \log_r n \rfloor^2}{n}, & \text{if } \alpha = 1, \\ (1+\varepsilon)\frac{C(\alpha,p)}{n^{(2-\alpha)/\alpha}}, & \text{if } 1 < \alpha < 2, \end{cases}$$

where the constant $C(\alpha,p)$ depends on the parameters α, p.

Finally, asymptotic expansions are established in [10] for the difference of the distribution functions $F_n^{\alpha,p}(\cdot) - G_{\alpha,p,\gamma_n}(\cdot)$ with uniform error terms depending on α.

Now we return to the problem of multiply Pauls. As in the preceding chapter a pooling strategy is an n dimensional vector $\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})$, such that $p_{1,n}, \dots, p_{n,n} \ge 0$ and $\sum_{k=1}^n p_{k,n} = 1$. Using this strategy, Paul_1 receives $p_{1,n}X_1 + p_{2,n}X_2 + \cdots + p_{n,n}X_n$ ducats, ..., Paul_n receives $p_{2,n}X_1 + p_{3,n}X_2 + \cdots + p_{1,n}X_n$ ducats. Assuming $\overline{p}_n = \max\{p_{1,n}, \dots, p_{n,n}\} \to 0$ for an infinite sequence of strategies $\{\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})\}$, our first interest in this chapter is the asymptotic distribution of

$$S_{\boldsymbol{p}_n}^{\alpha,p} = \sum_{k=1}^n p_{k,n}^{1/\alpha} X_k - \frac{p}{q} H_{\alpha,p}(\boldsymbol{p}_n), \tag{2.8}$$

a particular type of linear combinations when $\alpha \ne 1$, where

$$H_{\alpha,p}(\boldsymbol{p}_n) = \begin{cases} -\frac{1}{1-q^{\frac{1}{\alpha}-1}} \sum_{k=1}^n p_{k,n}^{1/\alpha}, & \text{if } \alpha \ne 1, \\ \sum_{k=1}^n p_{k,n} \log_r \frac{1}{p_{k,n}}, & \text{if } \alpha = 1. \end{cases}$$

Even though $p_{1,n}^{1/\alpha}, \dots, p_{n,n}^{1/\alpha}$ sum to one, and hence form a strategy only for $\alpha = 1$, it is a major technical step to come up with a merging approximation in terms of the distribution functions of the semistable random variables

$$W_{\boldsymbol{p}_n}^{\alpha,p} = \begin{cases} \sum_{k=1}^n p_{k,n}^{1/\alpha} W_{1,k}^{\alpha,p}, & \text{if } \alpha \ne 1, \\ \sum_{k=1}^n p_{k,n} W_{1,k}^{1,p} - \frac{p}{q} H_{1,p}(\boldsymbol{p}_n), & \text{if } \alpha = 1, \end{cases} \tag{2.9}$$

where the random variables $W_{1,1}^{\alpha,p}, W_{1,2}^{\alpha,p}, \ldots, W_{1,n}^{\alpha,p}$ are independent copies of $W_1^{\alpha,p}$, given by substituting $\gamma = 1$ in (2.2). The characteristic and the distribution functions will be denoted by $\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \mathbf{E}(\mathrm{e}^{\mathrm{i}tW_{\boldsymbol{p}_n}^{\alpha,p}})$ and $G_{\alpha,p,\boldsymbol{p}_n}(x) = \mathbf{P}\{W_{\boldsymbol{p}_n}^{\alpha,p} \le x\}$, $t, x \in \mathbb{R}$, respectively; the ostensible notational clash with (2.3), the strategy $\boldsymbol{p}_n$ appearing in place of γ, will turn out to be absolutely beneficial. It is easy to see that $W_{\boldsymbol{p}_n}^{\alpha,p}$ is indeed a semistable random variable with exponent α for an arbitrary strategy $\boldsymbol{p}_n$.

In the classical case, approximations of $\mathbf{P}\{S_{\boldsymbol{p}_n}^{1,1/2} \le x\}$ by $G_{1,1/2,\boldsymbol{p}_n}(x)$ were obtained in [17] with rates of merge. The main goal of the present paper is to generalize the merging asymptotic expansions in [10] to strategies, that is, to general linear combinations, such that the classical special case $\alpha = 1$, $p = 1/2$ of the expansion will yield the rates of merge in [17] and also show that those rates are not improvable. Our expansions here require certain mixed derivatives and their properties, which we now introduce, following [10] and [9]. Fix the parameters $\alpha \in (0, 2)$, $p \in (0, 1)$ and $\gamma \in (q, 1]$, and consider for each $u > 0$ the infinitely divisible distribution function $G_{\alpha,p,\gamma}(x\,; u)$, $x \in \mathbb{R}$, that has characteristic function $\mathbf{g}_{\alpha,p,\gamma}(t\,; u) = \mathrm{e}^{uy_\gamma^{\alpha,p}(t)}$, that is,

$$\mathbf{g}_{\alpha,p,\gamma}(t\,; u) = \mathrm{e}^{uy_\gamma^{\alpha,p}(t)} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}G_{\alpha,p,\gamma}(x\,; u), \quad t \in \mathbb{R}.$$

It was shown in Lemma 4 in [10] that the partial derivatives

$$\begin{aligned} G_{\alpha,p,\gamma}^{(k,j)}(x\,; u) &= \frac{\partial^{k+j}\, G_{\alpha,p,\gamma}(x\,; u)}{\partial x^k\, \partial u^j} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx} (-\mathrm{i}t)^{k-1} \big[y_\gamma^{\alpha,p}(t)\big]^j\, \mathrm{e}^{uy_\gamma^{\alpha,p}(t)}\,\mathrm{d}t \end{aligned} \tag{2.10}$$

are well defined at all $x \in \mathbb{R}$ and $u > 0$ for every $j \in \{0, 1, 2, \ldots\}$ and $k \in \mathbb{N}$, so that

$$G_{\alpha,p,\gamma}^{(k,j)}(x) = \frac{\partial^{k+j}\, G_{\alpha,p,\gamma}(x\,; u)}{\partial x^k\, \partial u^j}\bigg|_{u=1}, x \in \mathbb{R}, \text{ for } j \in \{0, 1, 2, \ldots\}, k \in \mathbb{N}, \tag{2.11}$$

are all meaningful. Furthermore, by Lemma 6 in [10] we have the moment property

$$\int_{-\infty}^{\infty} |x|^\beta \big|G_{\alpha,p,\gamma}^{(k+1,j)}(x)\big|\,\mathrm{d}x < \infty \quad 0 \le \beta < \alpha \text{ for all } j, k \in \{0, 1, 2, \ldots\}, \tag{2.12}$$

extending (2.5) from the case $G_{\alpha,p,\gamma}^{(1,0)}(\cdot) = G_{\alpha,p,\gamma}^{(2.1)}(\cdot) = G'_{\alpha,p,\gamma}(\cdot) = g_{\alpha,p,\gamma}(\cdot)$, and

$$G_{\alpha,p,\gamma}^{(k+1,j)}(\pm\infty) = \lim_{x\to\pm\infty} G_{\alpha,p,\gamma}^{(k+1,j)}(x) = 0 \text{ for all } j, k \in \{0, 1, 2, \ldots\}. \tag{2.13}$$

In particular, for every $j, k \in \{0, 1, 2, \ldots\}$ the function $G_{\alpha,p,\gamma}^{(k+1,j)}(\cdot)$ is Lebesgue integrable on $\mathbb{R}$, and hence

$$G_{\alpha,p,\gamma}^{(k,j)}(x) = \int_{-\infty}^{x} G_{\alpha,p,\gamma}^{(k+1,j)}(v)\,\mathrm{d}v, \quad x \in \mathbb{R},$$

is a function of bounded variation on the whole $\mathbb{R}$, with Fourier–Stieltjes transform

$$\begin{aligned} \mathbf{g}_{\alpha,p,\gamma}^{(k,j)}(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}G_{\alpha,p,\gamma}^{(k,j)}(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, G_{\alpha,p,\gamma}^{(k+1,j)}(x)\,\mathrm{d}x \\ &= (-\mathrm{i}t)^k \big[y_\gamma^{\alpha,p}(t)\big]^j \mathbf{g}_{\alpha,p,\gamma}(t) = (-\mathrm{i}t)^k \big[y_\gamma^{\alpha,p}(t)\big]^j \,\mathrm{e}^{y_\gamma^{\alpha,p}(t)}, \end{aligned} \tag{2.14}$$

for all $t \in \mathbb{R}$. These results in Lemma 6 in [10] are extended in [9] to arbitrary semistable distributions of exponent $\alpha \in (0, 2)$.

Theorem 2.1 in the next section contains the merging asymptotic expansions for the linear combinations in (2.8). However, these combinations are satisfactory for the n Pauls who wish to pool their individual winnings only in the case $\alpha = 1$. The equivalent Theorem 2.2 contains an overall satisfactory version after a simple transformation. As shown in [17] for $p = 1/2$ and in Theorem 1.1 (or in [26]) in general, for $\alpha = 1$ genuine benefits of pooling realize for a fixed n if and only if every component of the pooling strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ is either an integer power of $q = 1 - p$ or zero. Surprisingly, it will turn out in Corollary 2.2, that for any sequence of such *admissible* strategies there is a proper limiting distribution for $S_{\boldsymbol{p}_n}^{\alpha,p}$ and its equivalent form in Theorem 2.2 for every α, and the merging approximations reduce to asymptotic expansions of the usual type. The example of the *best admissible strategy* in [17] for the classical case $(\alpha, p) = (1, 1/2)$ is spelled out in detail. Numerical analysis is presented in Section 2.4., all the proofs are placed in Section 2.5..

2.3. The expansions

Fix any strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$, and consider the position parameters $\gamma_{k,n} = 1/(p_{k,n} r^{\lceil \log_r 1/p_{k,n} \rceil}) \in (q, 1]$ for each component $k = 1, 2, \ldots, n$ for which $p_{k,n} > 0$. Roughly speaking $\gamma_{k,n} \in (q, 1]$ determines the position of $p_{k,n}$ between two consecutive powers of r. Note that for the (generally inadmissible) uniform strategy $\boldsymbol{p}_n^{\diamond} = (1/n, \ldots, 1/n)$ all the $\gamma_{k,n}$ reduce to γ_n in (2.7). Recalling formula (2.3) for the ingredients and the notation $\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \mathbf{E}(\mathrm{e}^{\mathrm{i}tW_{\boldsymbol{p}_n}^{\alpha,p}})$ at (2.9), for $t \in \mathbb{R}$ we introduce the complex-valued function $\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)$, defined for $\alpha \neq 1$ as

$$\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) = \mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t)\Bigg[1 - \frac{1}{2}\sum_{k=1}^n p_{k,n}^2 \big[y_{\gamma_{k,n}}^{\alpha,p}(t)\big]^2 + \mathrm{i}ts_1^{\alpha,p}\sum_{k=1}^n p_{k,n}^{1+\frac{1}{\alpha}}\, y_{\gamma_{k,n}}^{\alpha,p}(t) + \frac{t^2}{2}\bigg\{(s_1^{\alpha,p})^2 + \frac{p}{q-q^{2/\alpha}}\bigg\}\sum_{k=1}^n p_{k,n}^{\frac{2}{\alpha}}\Bigg],$$

where the constant $s_1^{\alpha,p} = p/(q - q^{1/\alpha})$ is from (2.2), and for $\alpha = 1$ as

$$\mathbf{g}_{\boldsymbol{p}_n}^{1,p}(t) = \mathbf{g}_{1,p,\boldsymbol{p}_n}(t)\Bigg[1 - \frac{1}{2}\sum_{k=1}^n p_{k,n}^2 \big[y_{\gamma_{k,n}}^{1,p}(t)\big]^2 - \mathrm{i}t\frac{p}{q}\sum_{k=1}^n p_{k,n}^2\, y_{\gamma_{k,n}}^{1,p}(t)\log_r \frac{1}{p_{k,n}} + \frac{t^2}{2}\bigg\{\frac{p^2}{q^2}\sum_{k=1}^n p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}} + \frac{1}{q}\sum_{k=1}^n p_{k,n}^2\bigg\}\Bigg].$$

For any sequence $c_{1,n}, \ldots, c_{n,n}$ of complex numbers, where $c_{k,n}$ may be formally undefined if $p_{k,n} = 0$, here and throughout we use the convention $\sum_{k=1}^n p_{k,n}c_{k,n} = \sum_{\{1\le k\le n:\, p_{k,n}\neq 0\}} p_{k,n}c_{k,n}$. Consider finally the function $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ on $\mathbb{R}$ that has Fourier – Stieltjes transform $\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)$, that is,

$$\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, \mathrm{d}G_{\boldsymbol{p}_n}^{\alpha,p}(x), \quad t \in \mathbb{R}. \tag{2.15}$$

This is meaningful because the function $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ is a sum with four terms, the first of which is the distribution function $G_{\alpha,p,\boldsymbol{p}_n}(\cdot)$, while the other three terms will turn out to be constant multiples of sums of convolutions of well-determined distribution functions and some mixed derivatives in (2.11). To obtain an explicit formula of this nature for $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ we need the following scaling properties of the logarithm of the characteristic function in (2.4), which in particular will also be useful later for proving limit theorems for admissible strategies and which in general will add to our understanding in (2.16) below of the merging approximation itself.

For all $p \in (0,1)$ and $\gamma \in (q,1]$ the definition in (2.4) immediately implies

$$\gamma\, y_1^{\alpha,p}\bigg(\frac{t}{\gamma^{1/\alpha}}\bigg) = \begin{cases} y_\gamma^{\alpha,p}(t), & \text{if } \alpha \neq 1, \\ y_\gamma^{1,p}(t) - \mathrm{i}ts_\gamma^{1,p}, & \text{if } \alpha = 1, \end{cases} \qquad t \in \mathbb{R}.$$

Also, lengthy but straightforward calculation shows what in fact is the semistable property of the characteristic function $\mathbf{g}_{\alpha,p,\gamma}(\cdot)$ in (2.3), which for the classical case $(\alpha, p) = (1, 1/2)$ was first noticed by Martin-Löf ([31], Theorem 2), namely,

$$y_\gamma^{\alpha,p}\big(r^{\frac{m}{\alpha}} s\big) = \begin{cases} r^m y_\gamma^{\alpha,p}(s), & \text{if } \alpha \neq 1, \\ r^m y_\gamma^{1,p}(s) - \mathrm{i} s r^m m \frac{p}{q}, & \text{if } \alpha = 1, \quad s \in \mathbb{R}, \end{cases}$$

for all $m \in \mathbb{Z}$. Combining these two scaling properties we get for $\alpha \neq 1$,

$$y_1^{\alpha,p}\Big(t p_{k,n}^{1/\alpha}\Big) = p_{k,n}\, y_{\gamma_{k,n}}^{\alpha,p}(t), \quad t \in \mathbb{R},$$

and for $\alpha = 1$,

$$y_1^{1,p}(t p_{k,n}) = p_{k,n}\, y_{\gamma_{k,n}}^{1,p}(t) + \mathrm{i} t\, \frac{p}{q}\, p_{k,n} \log_r \frac{1}{p_{k,n}}, \quad t \in \mathbb{R}.$$

We claim that for all $\alpha \in (0,2)$, $p \in (0,1)$ and strategy $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ this implies the unified formula

$$\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \mathbf{E}\Big(\mathrm{e}^{\mathrm{i} t W_{\boldsymbol{p}_n}^{\alpha,p}}\Big) = \exp\left\{\sum_{k=1}^n p_{k,n}\, y_{\gamma_{k,n}}^{\alpha,p}(t)\right\}, \quad t \in \mathbb{R}, \tag{2.16}$$

for the pertaining characteristic functions. Indeed, if $\alpha \neq 1$, then

$$\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \prod_{k=1}^n \mathbf{g}_{\alpha,p,1}\Big(t p_{k,n}^{1/\alpha}\Big) = \prod_{k=1}^n \mathrm{e}^{y_1^{\alpha,p}\left(t p_{k,n}^{1/\alpha}\right)} = \exp\left\{\sum_{k=1}^n p_{k,n}\, y_{\gamma_{k,n}}^{\alpha,p}(t)\right\},$$

while if $\alpha = 1$, then

$$\begin{aligned} \mathbf{g}_{1,p,\boldsymbol{p}_n}(t) &= \mathrm{e}^{-\mathrm{i} t \frac{p}{q} H_{1,p}(\boldsymbol{p}_n)} \prod_{k=1}^n \mathbf{g}_{1,p,1}(t p_{k,n}) = \mathrm{e}^{-\mathrm{i} t \frac{p}{q} H_{1,p}(\boldsymbol{p}_n)} \prod_{k=1}^n \mathrm{e}^{y_1^{1,p}(t p_{k,n})} \\ &= \mathrm{e}^{-\mathrm{i} t \frac{p}{q} H_{1,p}(\boldsymbol{p}_n)} \exp\left\{\sum_{k=1}^n \left[p_{k,n}\, y_{\gamma_{k,n}}^{1,p}(t) + \mathrm{i} t\, \frac{p}{q}\, p_{k,n} \log_r \frac{1}{p_{k,n}}\right]\right\}, \end{aligned}$$

which, writing out the entropy $H_{1,p}(\boldsymbol{p}_n) = -\sum_{k=1}^n p_{k,n} \log_r p_{k,n}$, gives (2.16) also for $\alpha = 1$. Another general consequence of the scaling properties is that for all $\alpha \in (0,2)$ we can rewrite the functions $\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)$ in (2.15) in the following simpler form

$$\begin{aligned} \mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) = \mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t) \Bigg[1 &- \frac{1}{2} \sum_{k=1}^n \Big(y_1^{\alpha,p}\big(t p_{k,n}^{1/\alpha}\big)\Big)^2 + \mathrm{i} t s_1^{\alpha,p} \sum_{k=1}^n p_{k,n}^{1/\alpha}\, y_1^{\alpha,p}\big(t p_{k,n}^{1/\alpha}\big) \\ &+ \frac{t^2}{2} \left\{\big(s_1^{\alpha,p}\big)^2 + \frac{p}{q - q^{2/\alpha}}\right\} \sum_{k=1}^n p_{k,n}^{2/\alpha} \Bigg] \end{aligned} \tag{2.17}$$

for all $t \in \mathbb{R}$, noting also from (2.2) that $s_1^{1,p} = \frac{p}{q} \log 1 = 0$ for $\alpha = 1$.

Using the latter formula (2.17), we can now determine $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ as follows. The semistable random variable $p_{l,n}^{1/\alpha} W_1^{\alpha,p}$ has characteristic function

$$\mathbf{E}\Big(\mathrm{e}^{\mathrm{i}tp_{l,n}^{1/\alpha} W_1^{\alpha,p}}\Big) = \mathrm{e}^{y_1^{\alpha,p}(tp_{l,n}^{1/\alpha})}, \quad t \in \mathbb{R},$$

and distribution function

$$\mathbf{P}\Big\{p_{l,n}^{1/\alpha} W_1^{\alpha,p} \le x\Big\} = G_{\alpha,p,1}\big(xp_{l,n}^{-1/\alpha}\big), \quad x \in \mathbb{R},$$

for all $l = 1, 2, \ldots, n$ for which $p_{l,n} > 0$. Using (2.14) for $G_{\alpha,p,1}^{(m,j)}(x)$ and then replacing the latter argument x by $x/p_{l,n}^{1/\alpha}$, we obtain

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \mathrm{d}G_{m,j,l}^{\alpha,p,1}(x) = p_{l,n}^{m/\alpha}(-\mathrm{i}t)^m \left(y_1^{\alpha,p}\big(tp_{l,n}^{1/\alpha}\big)\right)^j \mathrm{e}^{y_1^{\alpha,p}(tp_{l,n}^{1/\alpha})}, \quad t \in \mathbb{R}, \tag{2.18}$$

where $G_{m,j,l}^{\alpha,p,1}(x) = G_{\alpha,p,1}^{(m,j)}(x/p_{l,n}^{1/\alpha})$, $x \in \mathbb{R}$, is of bounded variation, $m, j \ge 0$. Using (2.17) and the form

$$\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \mathrm{e}^{-I(\alpha=1)\mathrm{i}tp\, H_{1,p}(\boldsymbol{p}_n)/q} \prod_{k=1}^{n} \exp\left\{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})\right\}$$

from (2.9), where $I(A)$ is the indicator of the event A, for $\alpha \neq 1$ we obtain

$$\begin{aligned}
\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) = \mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t) &- \frac{1}{2}\sum_{k=1}^{n}\left[\left\{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})\right\}^2 \mathrm{e}^{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})} \prod_{\substack{j=1\\ j\neq k}}^{n} \mathrm{e}^{y_1^{\alpha,p}(tp_{j,n}^{1/\alpha})}\right] \\
&- s_1^{\alpha,p}\sum_{k=1}^{n}\left[p_{k,n}^{1/\alpha}(-\mathrm{i}t)y_1^{\alpha,p}\big(tp_{k,n}^{1/\alpha}\big)\, \mathrm{e}^{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})} \prod_{\substack{j=1\\ j\neq k}}^{n} \mathrm{e}^{y_1^{\alpha,p}(tp_{j,n}^{1/\alpha})}\right] \\
&- \frac{1}{2}\left[(s_1^{\alpha,p})^2 + \frac{p}{q - q^{2/\alpha}}\right]\sum_{k=1}^{n}\left[p_{k,n}^{2/\alpha}(-\mathrm{i}t)^2 \mathrm{e}^{y_1^{\alpha,p}(tp_{k,n}^{1/\alpha})}\prod_{\substack{j=1\\ j\neq k}}^{n} \mathrm{e}^{y_1^{\alpha,p}(tp_{j,n}^{1/\alpha})}\right]
\end{aligned} \tag{2.19}$$

for all $t \in \mathbb{R}$, and, setting $h_p(\boldsymbol{p}_n) = -pH_{1,p}(\boldsymbol{p}_n)/q$, for $\alpha = 1$,

$$\begin{aligned}
\mathbf{g}_{\boldsymbol{p}_n}^{1,p}(t) = \mathbf{g}_{1,p,\boldsymbol{p}_n}(t) &- \frac{\mathrm{e}^{\mathrm{i}th_p(\boldsymbol{p}_n)}}{2}\sum_{k=1}^{n}\left[\left\{y_1^{1,p}(tp_{k,n})\right\}^2 \mathrm{e}^{y_1^{1,p}(tp_{k,n})}\prod_{\substack{j=1\\ j\neq k}}^{n} \mathrm{e}^{y_1^{1,p}(tp_{j,n})}\right] \\
&- \frac{\mathrm{e}^{\mathrm{i}th_p(\boldsymbol{p}_n)}}{2\,q}\sum_{k=1}^{n}\left[p_{k,n}^2(-\mathrm{i}t)^2\, \mathrm{e}^{y_1^{1,p}(tp_{k,n})} \prod_{\substack{j=1\\ j\neq k}}^{n} \mathrm{e}^{y_1^{1,p}(tp_{j,n})}\right].
\end{aligned} \tag{2.20}$$

Consider the distribution functions $F_{k,n}^{\alpha,p}(x) = \mathbf{P}\big\{\sum_{j=1,j\neq k}^{n} p_{j,n}^{1/\alpha} W_{1,j}^{\alpha,p} \leq x\big\}$, $x \in \mathbb{R}$, where $W_{1,j}^{\alpha,p}$ are still independent copies of $W_1^{\alpha,p}$ in (2.2), $k = 1, \ldots, n$. Clearly, its characteristic function is

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, \mathrm{d}F_{k,n}^{\alpha,p}(x) = \prod_{\substack{j=1\\ j\neq k}}^{n} \mathrm{e}^{y_1^{\alpha,p}(tp_{j,n}^{1/\alpha})}, \quad t \in \mathbb{R}.$$

Using the notation $[F \star G](x) = \int_{-\infty}^{\infty} F(x-y)\,\mathrm{d}G(y) = \int_{-\infty}^{\infty} G(x-y)\,\mathrm{d}F(y)$, $x \in \mathbb{R}$, for the Lebesgue–Stieltjes convolution of the functions F and G of bounded variation and writing $s_1^{\alpha,p} = p/(q - q^{1/\alpha})$ in from (2.2), we see by (2.18) and (2.19) that for $\alpha \neq 1$,

$$\begin{aligned} G_{\boldsymbol{p}_n}^{\alpha,p}(x) = G_{\alpha,p,\boldsymbol{p}_n}(x) &- \frac{1}{2}\sum_{k=1}^{n}\Big[G_{0,2,k}^{\alpha,p,1} \star F_{k,n}^{\alpha,p}\Big](x) \\ &- \frac{p}{q - q^{1/\alpha}} \sum_{k=1}^{n}\Big[G_{1,1,k}^{\alpha,p,1} \star F_{k,n}^{\alpha,p}\Big](x) \\ &- \frac{1}{2}\bigg\{\frac{p^2}{(q-q^{1/\alpha})^2} + \frac{p}{q - q^{2/\alpha}}\bigg\} \sum_{k=1}^{n}\Big[G_{2,0,k}^{\alpha,p,1} \star F_{k,n}^{\alpha,p}\Big](x) \end{aligned} \tag{2.21}$$

and by (2.18) and (2.20) that for $\alpha = 1$,

$$\begin{aligned} G_{\boldsymbol{p}_n}^{1,p}(x) = G_{1,p,\boldsymbol{p}_n}(x) &- \frac{1}{2}\sum_{k=1}^{n}\Big[F_{h_p(\boldsymbol{p}_n)} \star G_{0,2,k}^{1,p,1} \star F_{k,n}^{1,p}\Big](x) \\ &- \frac{1}{2\,q}\sum_{k=1}^{n}\Big[F_{h_p(\boldsymbol{p}_n)} \star G_{2,0,k}^{1,p,1} \star F_{k,n}^{1,p}\Big](x) \end{aligned} \tag{2.22}$$

for all $x \in \mathbb{R}$, where $F_c(x) = 0$ or 1, according as $x < c$ or $x \geq c$, is the degenerate distribution function of the constant $c \in \mathbb{R}$.

The formulae (2.21) and (2.22) are very complicated and in fact useless to prove anything directly; for $\alpha = 1$ the expression (2.22) is even misleading in the sense that it does not contain the mixed derivative $G_{1,p,\gamma}^{(1,1)}(\cdot)$ for any $\gamma \in (q, 1]$. Nevertheless, they have two important consequences. One is the immediate fact that $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ is a function of bounded variation on $\mathbb{R}$, and hence (2.15) is indeed meaningful for all $\alpha \in (0,2)$, $p \in (0,1)$ and strategy $\boldsymbol{p}_n$. The other is that we see by (2.13) that to prove the important properties $G_{\boldsymbol{p}_n}^{\alpha,p}(-\infty) = 0$ and $G_{\boldsymbol{p}_n}^{\alpha,p}(\infty) = 1$ it suffices to show that $G_{\alpha,p,1}^{(0,2)}(\pm\infty) = 0$. This will be done in the next section, where, in turn, these properties are the key to get a numerically manageable formula for

$G^{\alpha,p}_{\boldsymbol{p}_n}(\cdot)$. We note that besides (2.21) and (2.22) intuitively more appealing formulae can be obtained directly by the defining formulae above (2.15) and by (2.16). Indeed, for any $u > 0$ introduce the functions $G^{(l+1,j)}_{u,\alpha,p,\gamma}(x) = G^{(l+1,j)}_{\alpha,p,\gamma}(x\,;u)$ in (2.10) and $G^{(l,j)}_{u,\alpha,p,\gamma}(x) = \int_{-\infty}^{x} G^{(l+1,j)}_{u,\alpha,p,\gamma}(y)\,\mathrm{d}y$, $x \in \mathbb{R}$, for which $\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}G^{(l,j)}_{u,\alpha,p,\gamma}(x) = (-\mathrm{i}t)^l [y^{\alpha,p}_{\gamma}(t)]^j\,\mathrm{e}^{u\,y^{\alpha,p}_{\gamma}(t)}$, $t \in \mathbb{R}$, by Lemma 6 in [10], $j, l \in \{0,1,2,\ldots\}$, which extends (2.14). Also, consider the semistable distribution function $H_{\alpha,p,k}(\cdot)$, which for any $k \in \{1,\ldots,n\}$ for which $p_{k,n} > 0$ is the convolution of $G_{p_{j,n},\alpha,p,\gamma_{j,n}}(\cdot)$ for all $j \in \{1,\ldots,n\}$, $j \neq k$, for which $p_{j,n} > 0$. Then for $\alpha = 1$,

$$\begin{aligned} G^{1,p}_{\boldsymbol{p}_n}(x) = G_{1,p,\boldsymbol{p}_n}(x) &- \sum_{k=1}^{n} \frac{p_{k,n}^2}{2} \Big[G^{(0,2)}_{p_{k,n},1,p,\gamma_{k,n}} \star H_{1,p,k} \Big](x) \\ &+ \frac{p}{q} \sum_{k=1}^{n} p_{k,n}^2 \left(\log_r \frac{1}{p_{k,n}} \right) \Big[G^{(1,1)}_{p_{k,n},1,p,\gamma_{k,n}} \star H_{1,p,k} \Big](x) \\ &- \left\{ \frac{p^2}{2q^2} \sum_{k=1}^{n} p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}} + \frac{1}{2q} \sum_{k=1}^{n} p_{k,n}^2 \right\} \Big[G^{(2,0)}_{p_{m,n},1,p,\gamma_{m,n}} \star H_{1,p,m} \Big](x) \end{aligned}$$

for all $x \in \mathbb{R}$, where $m \in \{1,\ldots,n\}$ is arbitrary as long as $p_{m,n} > 0$. It is easy to write down the analogous formula also for $\alpha \neq 1$.

Calculating directly from the corresponding special case of the formulae above (2.15), we point out right away for the uniform strategy $\boldsymbol{p}_n^{\diamond} = (1/n,\ldots,1/n)$ that by (2.14) and the fact — already noticed above — that $\gamma_{k,n} = \gamma_n$ in (2.7) for all $k = 1,\ldots,n$, so that $\mathbf{g}_{\alpha,p,\boldsymbol{p}_n^{\diamond}}(\cdot) = \mathbf{g}_{\alpha,p,\gamma_n}(\cdot)$ due to (2.16), we obtain

$$G^{\alpha,p}_{\boldsymbol{p}_n^{\diamond}}(x) = \begin{cases} G_{\alpha,p,\gamma_n}(x) - \frac{G^{(0,2)}_{\alpha,p,\gamma_n}(x)}{2\,n} - \frac{pG^{(1,1)}_{\alpha,p,\gamma_n}(x)}{\left(q - q^{\frac{1}{\alpha}}\right) n^{\frac{1}{\alpha}}} - \frac{p^2 G^{(2,0)}_{\alpha,p,\gamma_n}(x)}{2\left(q - q^{\frac{1}{\alpha}}\right)^2 n^{\frac{2-\alpha}{\alpha}}} - \frac{pG^{(2,0)}_{\alpha,p,\gamma_n}(x)}{2\left(q - q^{\frac{2}{\alpha}}\right) n^{\frac{2-\alpha}{\alpha}}}, \\ G_{1,p,\gamma_n}(x) - \frac{G^{(0,2)}_{1,p,\gamma_n}(x)}{2\,n} + \frac{pG^{(1,1)}_{1,p,\gamma_n}(x)\log_r n}{q\,n} - \frac{p^2 G^{(2,0)}_{1,p,\gamma_n}(x)\log_r^2 n}{2\,q^2\,n} - \frac{G^{(2,0)}_{1,p,\gamma_n}(x)}{2\,q\,n}, \end{cases}$$

for all $x \in \mathbb{R}$, where of course the upper branch is for $\alpha \neq 1$ and the lower branch is for $\alpha = 1$. For both branches the sum of the first four terms is the function $G^{\alpha,p}_{n}(x)$ in the Proposition in [10], where the fifth term was missed. That the inclusion of this fifth term would be a desirable adjustment in [10], at least for $\alpha \neq 1$, was noticed by Pap [34]. Hence for any strategy $\boldsymbol{p}_n$ the definition of $G^{\alpha,p}_{\boldsymbol{p}_n}(\cdot)$ in (2.15) is a suitable generalization of the desired full form $G^{\alpha,p}_{\boldsymbol{p}_n^{\diamond}}(\cdot)$ above. Then the main result for the merging approximation of the distribution function of $S^{\alpha,p}_{\boldsymbol{p}_n}$ from (2.8) is the following

Theorem 2.1. *For any sequence of strategies* $\{\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})\}_{n\in\mathbb{N}}$,

$$\sup_{x\in\mathbb{R}} \Big| \mathbf{P}\big\{ S_{\boldsymbol{p}_n}^{\alpha,p} \le x \big\} - G_{\boldsymbol{p}_n}^{\alpha,p}(x) \Big| = \begin{cases} O\big(\overline{p}_n^{\,2}\big), & \text{if } 0 < \alpha < 1/2, \\ O\big(\overline{p}_n^{\,1/\alpha}\big), & \text{if } 1/2 \le \alpha < 3/2; \\ O\big(\overline{p}_n^{\,(4-2\alpha)/\alpha}\big), & \text{if } 3/2 \le \alpha < 2, \end{cases}$$

where $\overline{p}_n = \max\{p_{1,n}, \dots, p_{n,n}\}$.

For the uniform strategy $\boldsymbol{p}_n^\diamond$, for which $S_{\boldsymbol{p}_n^\diamond}^{\alpha,p} = (S_n - c_n^{\alpha,p})/n^{1/\alpha}$ with S_n and $c_n^{\alpha,p}$ as in (2.6), Theorem 2.1 reduces to the Proposition in [10] when $\alpha \le 1$, with the adjusted full form of $G_{\boldsymbol{p}_n^\diamond}^{\alpha,p}(\cdot)$ replacing $G_n^{\alpha,p}(\cdot)$, except for a refined statement for non-lattice random variables in the case when $1/2 < \alpha < 1$. The real effect of the adjustment to $G_{\boldsymbol{p}_n^\diamond}^{\alpha,p}(\cdot)$ is for $\alpha \in (1,2)$, where the Proposition in [10] produces a worse rate for the approximation with $G_n^{\alpha,p}(\cdot)$ which precludes a real asymptotic expansion. In fact, for $\alpha \ne 1$ Pap [34] refined the result for $S_{\boldsymbol{p}_n^\diamond}^{\alpha,p}$ to a sort of a complete asymptotic expansion, the length of it is regulated by α: the closer α is to 0 or 2, the longer the expansion may be taken. As more refined statements than those in Theorem 2.1 and Theorem 2.2 below, we could have aimed at the generalization of his complete expansion to strategies, but we did not feel that the necessarily more complicated statements could give more insight into the problem, particularly that the more complicated terms of the approximation would be hard to penetrate for a reasonable interpretation. Finally we note that for $\alpha > 1$ Pap [34] proved the expansion for $S_{\boldsymbol{p}_n^\diamond}^{\alpha,p}$ in the stronger non-uniform form with the multiplicative factor $1 + |x|$. Again, we could have aimed at an analogous form here, multiplying the deviations in Theorems 2.1 and 2.2 by $1 + |x|$ before taking the supremum and keep the same order relations for $\alpha > 1$. However, in view of the tail behavior of the approximative distributions, for any given $\alpha \in (0,2)$ the useful result of this sort would be with the factor $1 + |x|^\alpha$. We conjecture that such non-uniform versions of Theorems 2.1 and 2.2 remain true; this would require new technical ideas and developments even for $\boldsymbol{p}_n^\diamond$.

As noted between (2.8) and (2.9), the sum of the weights $p_{1,n}^{1/\alpha}, \dots, p_{n,n}^{1/\alpha}$ in $S_{\boldsymbol{p}_n}^{\alpha,p}$ adds to unity only if $\alpha = 1$, so for $\alpha \ne 1$ they cannot represent a pooling strategy. Given these weights, we transform them to obtain a pooling strategy for arbitrary α in the following way. Let $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \dots, p_{n,n})$ be an arbitrary strategy as before and define $q_{j,n} = p_{j,n}^{1/\alpha} / \sum_{k=1}^n p_{k,n}^{1/\alpha}$, $j = 1, 2, \dots, n$. Then $\sum_{j=1}^n q_{j,n} = 1$, and so $\boldsymbol{q}_n = (q_{1,n}, q_{2,n}, \dots, q_{n,n})$ is also a strategy. In fact this is a one to one correspondence because, as can be seen easily, $p_{j,n} = q_{j,n}^\alpha / \sum_{k=1}^n q_{k,n}^\alpha$, $j = 1, 2, \dots, n$. Of course, for $\alpha = 1$ this is the identity correspondence. Using this transformation we can rewrite

Theorem 2.1 in an equivalent, more natural form. For an arbitrary strategy $\boldsymbol{q}_n = (q_{1,n}, \dots, q_{n,n})$, let

$$T_{\boldsymbol{q}_n}^{\alpha,p} = \frac{\sum_{k=1}^n q_{k,n} X_k}{\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{1/\alpha}} + \frac{p}{q - q^{1/\alpha}} \frac{1}{\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{1/\alpha}} \text{ and } V_{\boldsymbol{q}_n}^{\alpha,p} = \frac{\sum_{k=1}^n q_{k,n} W_{1,k}^{\alpha,p}}{\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{1/\alpha}}$$

if $\alpha \neq 1$, while $T_{\boldsymbol{q}_n}^{1,p} = S_{\boldsymbol{q}_n}^{1,p}$ and $V_{\boldsymbol{q}_n}^{1,p} = W_{\boldsymbol{q}_n}^{1,p}$ otherwise. Notice that $\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{1/\alpha}$ in the denominators is the ℓ_α-norm of the strategy $\boldsymbol{q}_n$. Also, for $\alpha \neq 1$ we introduce

$$\mathbf{h}_{\boldsymbol{q}_n}^{\alpha,p}(t) = \mathbf{E}\Big(\mathrm{e}^{\mathrm{i}tV_{\boldsymbol{q}_n}^{\alpha,p}}\Big)\left[1 - \frac{\sum_{k=1}^n q_{k,n}^{2\alpha}\left[y_{\nu_{k,n}}^{\alpha,p}(t)\right]^2}{2\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^2} + \frac{\mathrm{i}t s_1^{\alpha,p}\sum_{k=1}^n q_{k,n}^{1+\alpha} y_{\nu_{k,n}}^{\alpha,p}(t)}{\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{1+\frac{1}{\alpha}}}\right.$$
$$\left. + \frac{t^2\left((s_1^{\alpha,p})^2 + p/(q - q^{2/\alpha})\right)\sum_{k=1}^n q_{k,n}^2}{2\left(\sum_{j=1}^n q_{j,n}^\alpha\right)^{2/\alpha}}\right], \quad t \in \mathbb{R},$$

where $s_1^{\alpha,p} = p/(q-q^{1/\alpha})$ still and, again, just as for $\boldsymbol{p}_n$ above, the summations are only for those indices $k \in \{1, \dots, n\}$ for which $q_{k,n} > 0$, and for such k,

$$\nu_{k,n} = \frac{\frac{1}{q_{k,n}^\alpha}\sum_{j=1}^n q_{j,n}^\alpha}{r^{\left\lceil \log_r \frac{1}{q_{k,n}^\alpha}\sum_{j=1}^n q_{j,n}^\alpha \right\rceil}} \in (q, 1].$$

For $\alpha = 1$ we see that $\nu_{k,n}$ reduces to $\gamma_{k,n}$ that corresponds to $q_{k,n} > 0$, and we simply put $\mathbf{h}_{\boldsymbol{q}_n}^{1,p}(t) = \mathbf{g}_{\boldsymbol{q}_n}^{1,p}(t)$ for all $t \in \mathbb{R}$. Now consider the function $H_{\boldsymbol{q}_n}^{\alpha,p}(\cdot)$, of bounded variation on $\mathbb{R}$, that has Fourier–Stieltjes transform $\mathbf{h}_{\boldsymbol{q}_n}^{\alpha,p}(\cdot)$, so that $\mathbf{h}_{\boldsymbol{q}_n}^{\alpha,p}(t) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}H_{\boldsymbol{q}_n}^{\alpha,p}(x)$, $t \in \mathbb{R}$. Then we have the following

Theorem 2.2. *For any sequence of strategies* $\{\boldsymbol{q}_n = (q_{1,n}, \dots, q_{n,n})\}_{n\in\mathbb{N}}$,

$$\sup_{x\in\mathbb{R}}\left|\mathbf{P}\left\{T_{\boldsymbol{q}_n}^{\alpha,p} \le x\right\} - H_{\boldsymbol{q}_n}^{\alpha,p}(x)\right| = \begin{cases} O\big(h_{n,\alpha}^2\big), & \text{if } 0 < \alpha < 1/2, \\ O\big(h_{n,\alpha}^{1/\alpha}\big), & \text{if } 1/2 \le \alpha < 3/2, \\ O\big(h_{n,\alpha}^{(4-2\alpha)/\alpha}\big), & \text{if } 3/2 \le \alpha < 2, \end{cases}$$

where $h_{n,\alpha} = \overline{q}_n^\alpha / \sum_{k=1}^n q_{k,n}^\alpha$.

While formally these conditions are not required, Theorem 2.1 of course gives asymptotic results only when $\overline{p}_n \to 0$, while Theorem 2.2 works for a given α only if $h_{n,\alpha} \to 0$. This second condition is needed because, in general, the conditions $\overline{p}_n \to 0$ and $\overline{q}_n \to 0$ are independent in the sense that neither of them implies the other. This can be seen through suitably constructed examples.

Rates of merge with the distribution functions $H_{\alpha,p,\boldsymbol{q}_n}(x) = \mathbf{P}\{V_{\boldsymbol{q}_n}^{\alpha,p} \le x\}$, $x \in \mathbb{R}$, implying that in Theorem 4 in [17], are contained in the following

Corollary 2.1. *If* $\{\boldsymbol{q}_n = (q_{1,n}, \dots, q_{n,n})\}_{n\in\mathbb{N}}$ *is a sequence of strategies for which* $h_{n,\alpha} \to 0$, *then for every* $\varepsilon > 0$ *there is a threshold* $n_* = n_*(\varepsilon, \alpha, p) \in \mathbb{N}$ *such that*

$$\sup_{x\in\mathbb{R}} \left|\mathbf{P}\left\{T_{\boldsymbol{q}_n}^{\alpha,p} \le x\right\} - H_{\alpha,p,\boldsymbol{q}_n}(x)\right| \le \begin{cases} (1+\varepsilon)\, K(\alpha,p)\, h_{n,\alpha}, & \text{if } 0<\alpha<1, \\ (1+\varepsilon)\, K(1,p)\, \overline{q}_n \log_r^2 \frac{1}{\overline{q}_n}, & \text{if } \alpha = 1, \\ (1+\varepsilon)\, K(\alpha,p)\, h_{n,\alpha}^{(2-\alpha)/\alpha}, & \text{if } 1<\alpha<2, \end{cases}$$

whenever $n \ge n_*$, *where the constants are*

$$K(\alpha,p) = \begin{cases} \frac{C_7^2}{2\pi\alpha C_1^2}, & \text{if } 0<\alpha<1, \\ \frac{p^2}{2q^2\pi C_1^2}, & \text{if } \alpha=1, \\ \left\{\frac{p^2}{(q-q^{1/\alpha})^2} + \frac{p}{q-q^{2/\alpha}}\right\} \frac{\Gamma(2/\alpha)}{2\pi\alpha C_1^{2/\alpha}}, & \text{if } 1<\alpha<2, \end{cases}$$

where $\Gamma(u) = \int_0^\infty x^{u-1}\, \mathrm{e}^{-x}\, \mathrm{d}x$, $u>0$, *is the usual gamma function, in which*

$$C_1 = C_1(\alpha,p) = \left(\frac{2}{\pi}\right)^\alpha \frac{pq^{(2-\alpha)/\alpha}}{q - q^{2/\alpha}}$$

and, for $\alpha < 1$,

$$C_7 = C_7(\alpha,p) = \frac{2^{1-\alpha}}{q} + \frac{2^{1-\alpha}p}{q - q^{1/\alpha}}.$$

The admissibility condition is difficult to formulate in the context of the $\boldsymbol{q}_n$ weights of Theorem 2.2, so in this regard we focus only on Theorem 2.1. Since all nonzero members $p_{k,n}$ of an admissible strategy are integer powers of q, the corresponding $\gamma_{k,n} = 1$, $k = 1,2,\dots,n$. Hence by (2.16) for any admissible strategy $\boldsymbol{p}_n$ the distributional equality $W_{\boldsymbol{p}_n}^{\alpha,p} \stackrel{\mathcal{D}}{=} W_1^{\alpha,p}$ holds for the random variable $W_1^{\alpha,p}$ in (2.2), and the functions $\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)$ in (2.15) may be written in the following simpler form: for $\alpha \neq 1$,

$$\begin{aligned} \mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t) = \mathrm{e}^{y_1^{\alpha,p}(t)} - \left[y_1^{\alpha,p}(t)\right]^2 \mathrm{e}^{y_1^{\alpha,p}(t)} \frac{1}{2}\sum_{k=1}^n p_{k,n}^2 - (-\mathrm{i}t) y_1^{\alpha,p}(t)\, \mathrm{e}^{y_1^{\alpha,p}(t)} \frac{p\sum_{k=1}^n p_{k,n}^{1+\frac{1}{\alpha}}}{q - q^{1/\alpha}} \\ - (-\mathrm{i}t)^2 \mathrm{e}^{y_1^{\alpha,p}(t)} \left\{\frac{p^2}{(q-q^{1/\alpha})^2} + \frac{p}{q-q^{2/\alpha}}\right\} \frac{1}{2}\sum_{k=1}^n p_{k,n}^{2/\alpha}, \end{aligned}$$

and for $\alpha = 1$,

$$\mathbf{g}^{1,p}_{\boldsymbol{p}_n}(t) = \mathrm{e}^{y_1^{1,p}(t)} - \big[y_1^{1,p}(t)\big]^2\,\mathrm{e}^{y_1^{1,p}(t)}\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^2 - \mathrm{i}t y_1^{1,p}(t)\,\mathrm{e}^{y_1^{1,p}(t)}\frac{p}{q}\sum_{k=1}^{n} p_{k,n}^2 \log_r \frac{1}{p_{k,n}}$$
$$- \frac{(-\mathrm{i}t)^2 \mathrm{e}^{y_1^{1,p}(t)}}{2}\bigg\{\frac{p^2}{q^2}\sum_{k=1}^{n} p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}} + \frac{1}{q}\sum_{k=1}^{n} p_{k,n}^2\bigg\}.$$

Thus for any admissible strategy $\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})$ by (2.14) we have for $\alpha \neq 1$,

$$G^{\alpha,p}_{\boldsymbol{p}_n}(x) = G_{\alpha,p,1}(x) - G^{(0,2)}_{\alpha,p,1}(x)\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^2 - G^{(1,1)}_{\alpha,p,1}(x)\frac{p}{q - q^{1/\alpha}}\sum_{k=1}^{n} p_{k,n}^{1+\frac{1}{\alpha}}$$
$$- G^{(2,0)}_{\alpha,p,1}(x)\bigg\{\frac{p^2}{(q - q^{1/\alpha})^2} + \frac{p}{q - q^{2/\alpha}}\bigg\}\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^{\frac{2}{\alpha}},$$

and for $\alpha = 1$,

$$G^{1,p}_{\boldsymbol{p}_n}(x) = G_{1,p,1}(x) - G^{(0,2)}_{1,p,1}(x)\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^2 + G^{(1,1)}_{1,p,1}(x)\frac{p}{q}\sum_{k=1}^{n} p_{k,n}^2 \log_r \frac{1}{p_{k,n}}$$
$$- G^{(2,0)}_{1,p,1}(x)\frac{1}{2}\bigg\{\frac{p^2}{q^2}\sum_{k=1}^{n} p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}} + \frac{1}{q}\sum_{k=1}^{n} p_{k,n}^2\bigg\} \tag{2.23}$$

for all $x \in \mathbb{R}$. Therefore, in the admissible case there exists a proper limiting distribution, and moreover we have real asymptotic expansions attached to this asymptotic distribution. Concentrating on the dominant terms in Theorem 2.1, we obtain the following

Corollary 2.2. *For any sequence $\{\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})\}_{n\in\mathbb{N}}$ of admissible strategies, for $\alpha \in (0,1)$,*

$$\sup_{x\in\mathbb{R}}\bigg|\mathbf{P}\big\{S^{\alpha,p}_{\boldsymbol{p}_n} \le x\big\} - \bigg[G_{\alpha,p,1}(x) - G^{(0,2)}_{\alpha,p,1}(x)\frac{1}{2}\sum_{k=1}^{n} p_{k,n}^2\bigg]\bigg|$$
$$= \begin{cases} O(\overline{p}_n^2), & \text{if } 0 < \alpha \le 1/2, \\ O(\overline{p}_n^{1/\alpha}), & \text{if } 1/2 < \alpha < 1; \end{cases}$$

for $\alpha = 1$,

$$\sup_{x\in\mathbb{R}}\bigg|\mathbf{P}\big\{S^{1,p}_{\boldsymbol{p}_n} \le x\big\} - \bigg[\ G_{1,p,1}(x) + G^{(1,1)}_{1,p,1}(x)\frac{p}{q}\sum_{k=1}^{n} p_{k,n}^2 \log_r \frac{1}{p_{k,n}}$$
$$- G^{(2,0)}_{1,p,1}(x)\frac{p^2}{2q^2}\sum_{k=1}^{n} p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}}\bigg]\bigg| = O(\overline{p}_n);$$

and for $\alpha \in (1,2)$,

$$\sup_{x\in\mathbb{R}}\left|\mathbf{P}\{S_{\boldsymbol{p}_n}^{\alpha,p}\le x\}-\left[G_{\alpha,p,1}(x)-G_{\alpha,p,1}^{(2,0)}(x)\left\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\right\}\frac{1}{2}\sum_{k=1}^{n}p_{k,n}^{2/\alpha}\right]\right|$$
$$=\begin{cases} O(\overline{p}_n^{1/\alpha}), & \text{if } 1<\alpha\le 3/2,\\ O(\overline{p}_n^{(4-2\alpha)/\alpha}), & \text{if } 3/2<\alpha<2.\end{cases}$$

For each $n \in \mathbb{N}$ the best admissible strategy for the classical case $(\alpha, p) = (1, 1/2)$, found in [17], is the following:

$$\boldsymbol{p}_n^* = (p_{1,n}^*,\ldots,p_{n,n}^*) = (2p_n^*,\ldots,2p_n^*,p_n^*,\ldots,p_n^*) \quad \text{with} \quad p_n^* = \frac{1}{2^{\lceil \log_2 n\rceil}} = \frac{\gamma_n}{n},$$

where the number of the p_n^* components is $2n-2^{\lceil \log_2 n\rceil}$ and the number of the $2p_n^*$ components is $2^{\lceil \log_2 n\rceil}-n$. Calculating the coefficients in (2.23), we obtain $G_{\boldsymbol{p}_n^*}^{1,1/2}(x) = G_{1,1/2,1}(x) - a_n G_{1,1/2,1}^{(0,2)}(x) + b_n G_{1,1/2,1}^{(1,1)}(x) - c_n G_{1,1/2,1}^{(2,0)}(x)$, $x \in \mathbb{R}$, and $\sup_{x\in\mathbb{R}}\bigl|\mathbf{P}\{S_{\boldsymbol{p}_n^*}^{1,1/2}\le x\}-\bigl[G_{1,1/2,1}(x)+b_nG_{1,1/2,1}^{(1,1)}(x)-d_nG_{1,1/2,1}^{(2,0)}(x)\bigr]\bigr| = O(1/n)$ as a special case of Corollary 2.2, where $\gamma_n = n/2^{\lceil \log_2 n\rceil}$ oscillates in $(1/2, 1]$,

$$\begin{aligned}
a_n &= \frac{3\cdot 2^{\lceil \log_2 n\rceil}-2n}{2^{2\lceil \log_2 n\rceil+1}} = \frac{\frac{3}{2}\gamma_n-\gamma_n^2}{n},\\
b_n &= \frac{(3\cdot 2^{\lceil \log_2 n\rceil}-2n)\lceil \log_2 n\rceil - 4(2^{\lceil \log_2 n\rceil}-n)}{2^{2\lceil \log_2 n\rceil}}\\
&= \frac{(3\gamma_n-2\gamma_n^2)\log_2\frac{n}{\gamma_n}-4(\gamma_n-\gamma_n^2)}{n},\\
d_n &= \frac{(3\cdot 2^{\lceil \log_2 n\rceil}-2n)\lceil \log_2 n\rceil^2-4(2^{\lceil \log_2 n\rceil}-n)(2\lceil \log_2 n\rceil-1)}{2^{2\lceil \log_2 n\rceil+1}}\\
&= \frac{(\frac{3}{2}\gamma_n-\gamma_n^2)\log_2^2\frac{n}{\gamma_n}-2(\gamma_n-\gamma_n^2)(2\log_2 n\gamma_n-1)}{n},
\end{aligned}$$

and

$$c_n = d_n + \frac{6\cdot 2^{\lceil \log_2 n\rceil}-4n}{2^{2\lceil \log_2 n\rceil+1}} = d_n + \frac{3\gamma_n-2\gamma_n^2}{n}.$$

Also, since in the proof of Corollary 2.1 we show for all $p \in (0, 1)$ and all $\{\boldsymbol{p}_n\}$ for which $\overline{p}_n \to 0$ that for every $\varepsilon > 0$ there is an $n_*(\varepsilon, p) \in \mathbb{N}$ such that for $n \ge n_*(\varepsilon, p)$,

$$\sup_{x\in\mathbb{R}}\left|\mathbf{P}\left\{S_{\boldsymbol{p}_n}^{1,p}\le x\right\}-G_{1,p,\boldsymbol{p}_n}(x)\right| \le (1+\varepsilon)\frac{p^2}{q^2\pi C_1^2}\sum_{k=1}^{n}\frac{p_{k,n}^2}{2}\log_r^2\frac{1}{p_{k,n}},$$

and since the last sum for $\boldsymbol{p}_n = \boldsymbol{p}_n^*$ is exactly d_n, for which the asymptotic equality

$$d_n \sim \frac{\gamma_n(3-2\gamma_n)}{2} \frac{\log_2^2 n}{n},$$

is satisfied, where we write $x_n \sim y_n$ if $x_n / y_n \to 1$, substituting $C_1(1, 1/2) = 2/\pi$ we obtain

$$\sup_{x \in \mathbb{R}} \left| \mathbf{P}\left\{ S_{\boldsymbol{p}_n^*}^{1,1/2} \leq x \right\} - G_{1,1/2,1}(x) \right| \leq (1+\varepsilon) \frac{\pi \gamma_n (3-2\gamma_n)}{8} \frac{\log_2^2 n}{n}$$

whenever $n \geq n_*(\varepsilon)$, a slightly better bound than the one in (34) in [17].

2.4. Numerical computations

The merging semistable approximations are described only through their characteristic functions and their mathematical properties are inferred either through Fourier-analytic methods or by special representations, such as that in (2.2). The same is even more true for the derivatives featured in our expansions, for which the only conceivable tool appears to be the Fourier method. For the purpose of numerical investigation of the expansions we use what we call the *extended Gil-Pelaez – Rosén inversion formula*, which says the following. Let $H(\cdot)$ be a function of bounded variation on $\mathbb{R}$, consider its total variation function $V_H(x) = \sup\{\sum_{j=1}^n |H(x_j) - H(x_{j-1})| : -\infty < x_0 < x_1 < \cdots < x_n \leq x, n \in \mathbb{N}\}$ and let $\mathbf{h}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \mathrm{d}H(x)$ be its Fourier – Stieltjes transform, $t \in \mathbb{R}$. If the logarithmic moment $\int_{-\infty}^{\infty} \log(1 + |x|)\, \mathrm{d}V_H(x) < \infty$, then

$$\frac{H(x+0) - H(x-0)}{2} = \frac{H(\infty) - H(-\infty)}{2} - \frac{1}{\pi} \lim_{T \to \infty} \int_0^T \frac{\mathfrak{Im}\left\{\mathrm{e}^{-\mathrm{i}tx} \mathbf{h}(t)\right\}}{t} \mathrm{d}t$$

for every $x \in \mathbb{R}$, where $H(\pm\infty) = \lim_{x \to \pm\infty} H(x)$. Gil-Pelaez [22] proved this for distribution functions without the logarithmic moment condition, in which case the integral is also improper Riemann at zero. Eleven years later Rosén [36] independently proved the same formula also for a distribution function H, for which $H(\infty) - H(-\infty) = 1 - 0 = 1$, showing in particular that under the logarithmic moment condition the integral exists as a proper Lebesgue integral on $(0, T]$ for all $T > 0$. A trivial modification of Rosén's proof gives the extended form above.

The Gil-Pelaez – Rosén formula is clearly applicable to the distribution function $G_{\alpha,p,\boldsymbol{p}_n}(\cdot)$. In order to use the formula for $G_{\boldsymbol{p}_n}^{\alpha,p}(\cdot)$ we claim that $G_{\boldsymbol{p}_n}^{\alpha,p}(\infty) = 1$ and $G_{\boldsymbol{p}_n}^{\alpha,p}(-\infty) = 0$ for every $\alpha \in (0,2)$, $p \in (0,1)$ and strategy

$\boldsymbol{p}_n$. As already noted in the previous section, by (2.13), (2.21) and (2.22) it suffices to show that $G_{\alpha,p,\gamma}^{(0,2)}(\pm\infty) = 0$ for all $\gamma \in (q, 1]$. We know that $G_{\alpha,p,\gamma}^{(0,2)}(x) = \int_{-\infty}^{x} G_{\alpha,p,\gamma}^{(1,2)}(y)\mathrm{d}y$, $x \in \mathbb{R}$, for the integrable function $G_{\alpha,p,\gamma}^{(1,2)}(\cdot)$, thus $G_{\alpha,p,\gamma}^{(0,2)}(-\infty) = 0$ and $G_{\alpha,p,\gamma}^{(0,2)}(\cdot)$ is of bounded variation. The logarithmic moment property also holds by (2.12), hence by the extended Gil-Pelaez – Rosén formula

$$G_{\alpha,p,\gamma}^{(0,2)}(x) = \frac{G_{\alpha,p,\gamma}^{(0,2)}(\infty)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathfrak{Im}\left\{\mathrm{e}^{-\mathrm{i}tx}\left[y_\gamma^{\alpha,p}(t)\right]^2 \mathrm{e}^{y_\gamma^{\alpha,p}(t)}\right\}}{t}\,\mathrm{d}t, \quad x \in \mathbb{R},$$

where we write the integral in this proper form since by Lemma 2.2 below the function $t \mapsto [y_\gamma^{\alpha,p}(t)]^2\,\mathrm{e}^{y_\gamma^{\alpha,p}(t)}/t$ is in fact Lebesgue integrable on $(0,\infty)$. Thus the Riemann – Lebesgue lemma implies that $G_{\alpha,p,\gamma}^{(0,2)}(\infty) = G_{\alpha,p,\gamma}^{(0,2)}(\infty)/2$, and hence $G_{\alpha,p,\gamma}^{(0,2)}(\infty) = 0$ indeed. We note that the same argument shows that $G_\alpha^{(k,j)}(\infty) = 0$ for all $k, j = 0, 1, \ldots$ for which $k + j > 0$ for any semistable distribution function $G_\alpha(\cdot)$ with characteristic exponent $\alpha \in (0, 2)$; these derivatives are developed in [9].

Now, applying the extended Gil-Pelaez – Rosén formula, we obtain

$$G_{\boldsymbol{p}_n}^{\alpha,p}(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathfrak{Im}\left\{\mathrm{e}^{-\mathrm{i}tx}\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)\right\}}{t}\,\mathrm{d}t, \quad x \in \mathbb{R}.$$

Due to the mass concentrating near zero, this formula is numerically inconvenient. The problem can be overcome by the change of variables $t = \mathrm{e}^u$, which gives

$$G_{\boldsymbol{p}_n}^{\alpha,p}(x) = \frac{1}{2} - \frac{1}{\pi}\int_{-\infty}^\infty \mathfrak{Im}\left\{\mathrm{e}^{-\mathrm{i}x\mathrm{e}^u}\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(\mathrm{e}^u)\right\}\mathrm{d}u, \quad x \in \mathbb{R},$$

and smears that mass on the whole negative half-line. Indeed, using Simpson's method for numerical integration, we found that for all values of the parameters and for all strategies considered in the examples below it suffices to integrate on the finite interval $[-20, 3]$. The idea of transforming variables and the whole computation for the distribution functions $G_{1,1/2,\gamma}(\cdot)$ is due to Gordon Simons. The exact same formula can be shown to produce $H_{\boldsymbol{q}_n}^{\alpha,p}(\cdot)$ from $\mathbf{h}_{\boldsymbol{q}_n}^{\alpha,p}(\cdot)$ in the context of Theorem 2.2.

It was with this method that the three examples in Figures 1–3 in [10] were obtained for the uniform averaging strategy $\boldsymbol{p}_n^\diamond = (1/n, \ldots, 1/n)$ for $\alpha = 3/2, 1, 1/2$ and the respective $n = 50, 10, 7$, all with $p = 1/2$. For the six examples here we chose the same α parameters with some different, but still very small n. Figures $1, 2, 3, 6$ are for the choices $\alpha = 3/2, 1, 1, 1/2$ and the

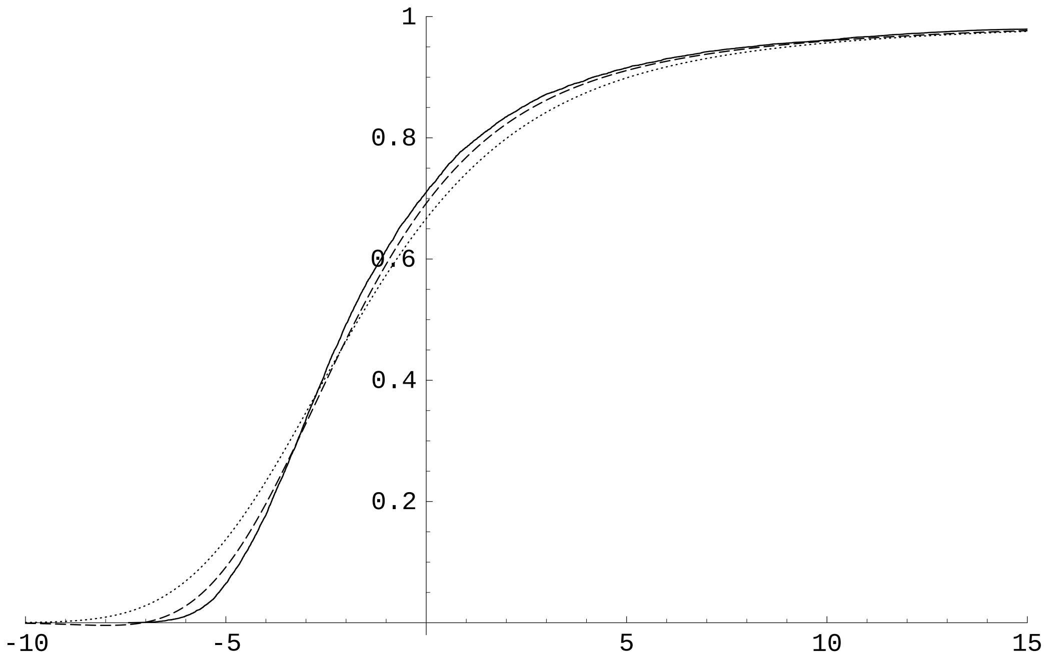

Figure 1. Solid $F^{3/2,1/2}_{\boldsymbol{p}_{100}}$, dotted $G_{3/2,1/2,\boldsymbol{p}_{100}}$, and dashed $G^{3/2,1/2}_{\boldsymbol{p}_{100}}$

strategies

$$\boldsymbol{p}_{100} = \Big(\underbrace{\frac{1}{80},\ldots,\frac{1}{80}}_{40\text{ times}},\underbrace{\frac{1}{120},\ldots,\frac{1}{120}}_{60\text{ times}}\Big), \quad \boldsymbol{p}^*_{12} = \Big(\underbrace{\frac{1}{8},\ldots,\frac{1}{8}}_{4\text{ times}},\underbrace{\frac{1}{16},\ldots,\frac{1}{16}}_{8\text{ times}}\Big),$$

$$\boldsymbol{p}_{12} = \Big(\underbrace{\frac{1}{10},\ldots,\frac{1}{10}}_{6\text{ times}},\underbrace{\frac{1}{15},\ldots,\frac{1}{15}}_{6\text{ times}}\Big), \quad \boldsymbol{p}_{8} = \Big(\underbrace{\frac{1}{6},\ldots,\frac{1}{6}}_{4\text{ times}},\underbrace{\frac{1}{12},\ldots,\frac{1}{12}}_{4\text{ times}}\Big),$$

respectively; in these four cases we still chose the unbiased situation of historical interest, that is, $p = 1/2$. For the most interesting case $\alpha = 1$ of the tail or payoff parameter, for which the mean becomes infinite, we also investigated the dependence of the approximation on the bias parameter p: with $\boldsymbol{p}_{12}$ kept, Figures 4 and 5 are for the choices $p = 1/10$ and $p = 5/6$. On all six figures the solid curves depict the distribution functions $F^{\alpha,p}_{\boldsymbol{p}_n}(x) = \mathbf{P}\{S^{\alpha,p}_{\boldsymbol{p}_n} \le x\}$, $x \in \mathbb{R}$, which are obtained as the empirical distribution functions of 10 000 simulations of $S^{\alpha,p}_{\boldsymbol{p}_n}$. Also on all six figures the dotted curves $G_{\alpha,p,\boldsymbol{p}_n}(\cdot)$ are the merging semistable distribution functions and the dashed curves are the full approximations $G^{\alpha,p}_{\boldsymbol{p}_n}(\cdot)$ of Theorem 2.1.

In Figure 1. $\alpha = 3/2$, thus the rate of merge is $\overline{p}_n^{1/3}$ and the order of the approximation is $\overline{p}_n^{2/3}$. The very satisfactory full approximation provides a dramatic improvement.

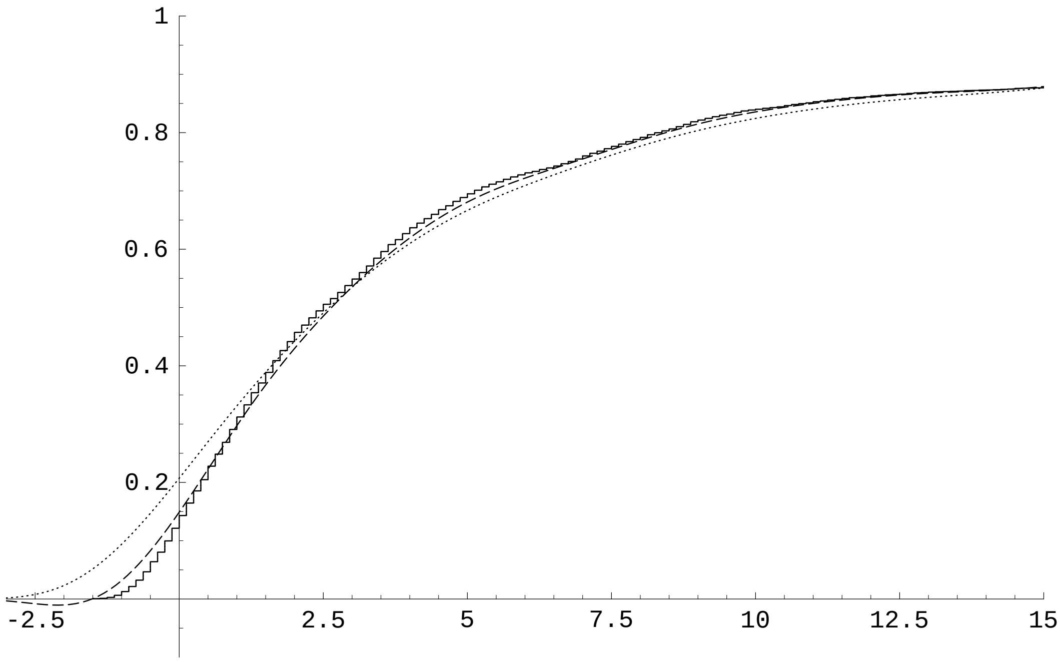

Figure 2. Solid $F^{1,1/2}_{\boldsymbol{p}^*_{12}}$, dotted $G_{1,1/2,\boldsymbol{p}^*_{12}}$, and dashed $G^{1,1/2}_{\boldsymbol{p}^*_{12}}$

For $\alpha = 1$ the rate of merge is $\overline{p}_n \log_2^2 1/\overline{p}_n$ and the order of the approximation is $\overline{p}_n$. The best admissible strategy can be seen in Figure 2., for which $G_{1,1/2,\boldsymbol{p}^*_n}(\cdot) \equiv G_{1,1/2,1}(\cdot)$ for all n. The example in the next figure is the exact opposite of this, but no particular difference is visible. The two different values of $\gamma_{k,12}$, $k = 1, 2, \ldots, 12$, for the strategy here, 10/16 and 15/16, differ from each other to a great extent. Roughly speaking this means that $G_{1,1/2,\boldsymbol{p}_{12}}(\cdot)$ differs from a single distribution function $G_{1,1/2,\gamma}(\cdot)$, for any γ, as much as it can. But the quality of the approximation is about the same as in Figure 2.

In Figures 4. and 5. we illustrate the dependence on p. In both cases $\alpha = 1$. In Figure 4. $p = 1/10$ thus $r = 10/9$, so that the gains, the powers of 10/9, increase very slowly. An easy computation shows that $F^{1,1/10}_{\boldsymbol{p}_n}(\cdot)$ has about $8 \cdot 10^{11}$ jump points in the interval $(-3, 15)$, so it seems to be continuous. As the following figure shows, for a large p the situation is the opposite. In this case $r = 6$, so the gains increase very fast. One can easily count that $F^{1,5/6}_{\boldsymbol{p}_n}(\cdot)$ has 20 jump points in $(-3, 20)$. Thus n ought to be larger here to obtain a better approximation.

In the last figure, for $\alpha = 1/2$ the rate of merge is $\overline{p}_n$, while the order of the full approximation is much better, $\overline{p}_n^2$. The precision is almost unbelievably good for even $n = 8$. We also note that despite the fact that the present strategy is not admissible, we still have $\gamma_{k,8} \equiv 3/4$ for all $k = 1, 2, \ldots, 8$, so

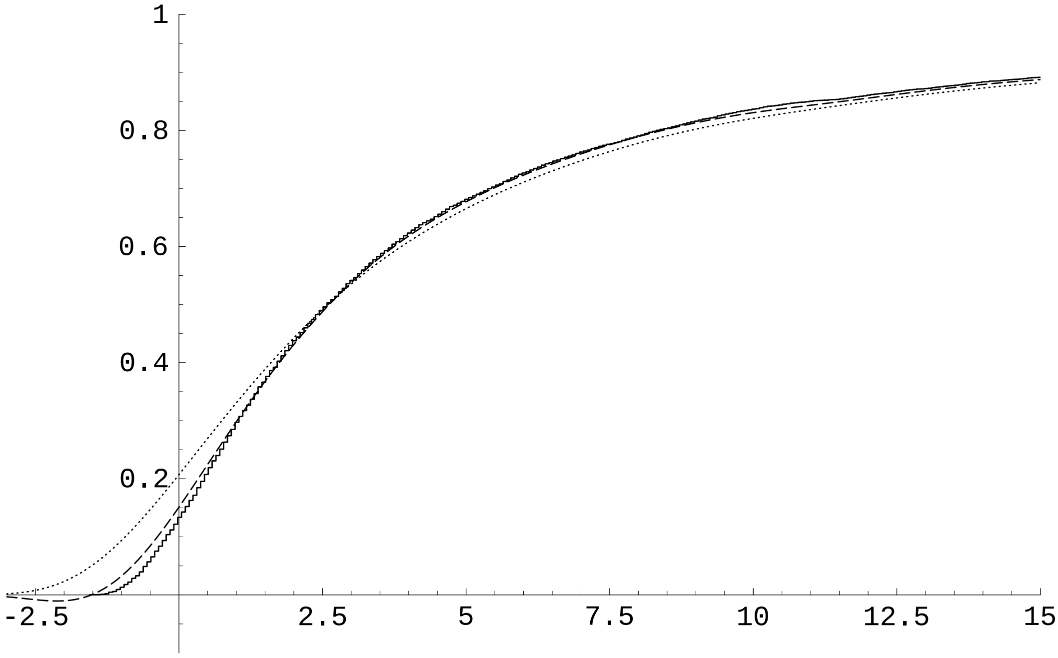

Figure 3. Solid $F^{1,1/2}_{\boldsymbol{p}_{12}}$, dotted $G_{1,1/2,\boldsymbol{p}_{12}}$, and dashed $G^{1,1/2}_{\boldsymbol{p}_{12}}$

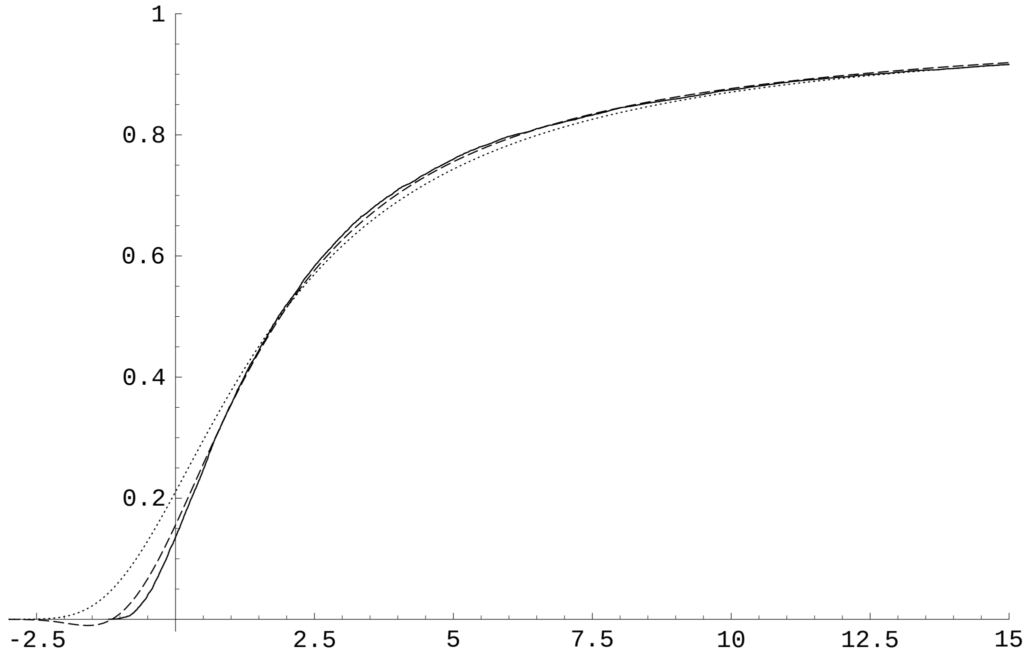

Figure 4. Solid $F^{1,1/10}_{\boldsymbol{p}_{12}}$, dotted $G_{1,1/10,\boldsymbol{p}_{12}}$, and dashed $G^{1,1/10}_{\boldsymbol{p}_{12}}$

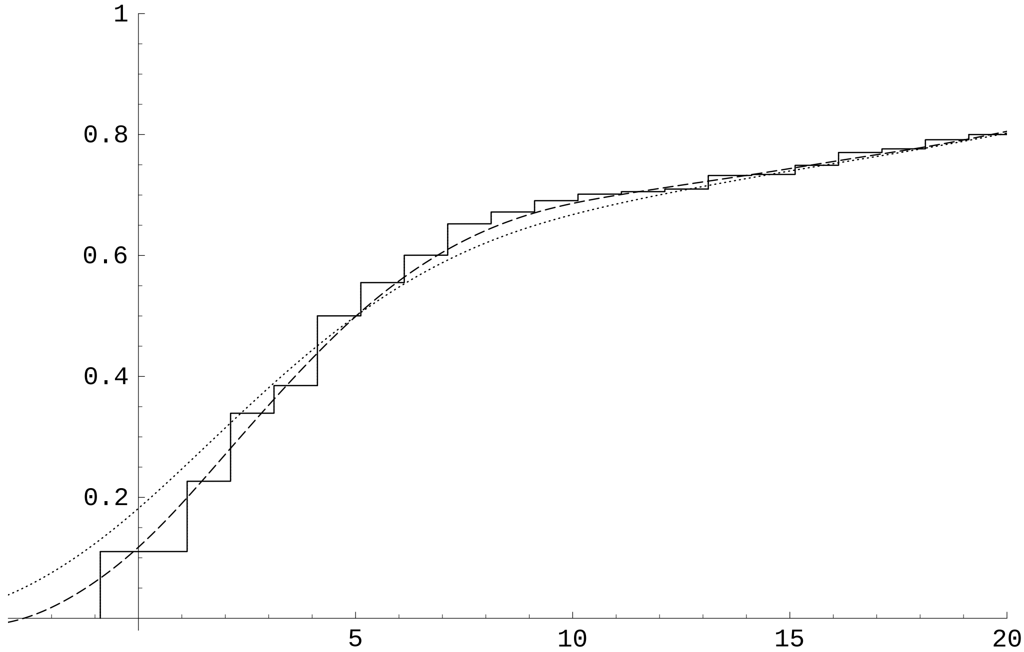

Figure 5. Solid $F^{1,5/6}_{\boldsymbol{p}_{12}}$, dotted $G_{1,5/6,\boldsymbol{p}_{12}}$, and dashed $G^{1,5/6}_{\boldsymbol{p}_{12}}$

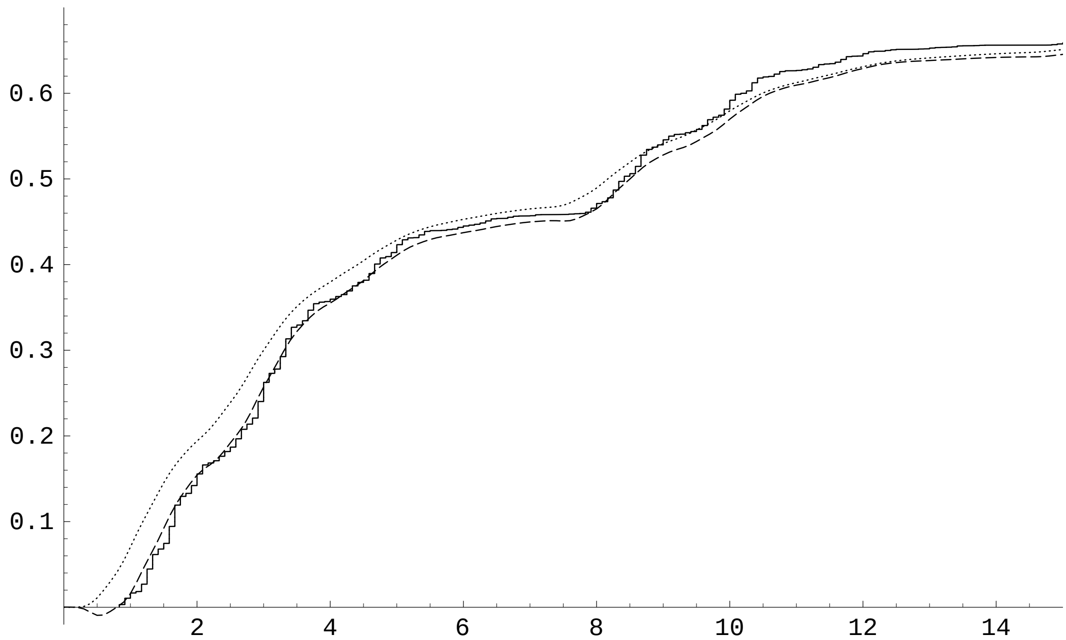

Figure 6. Solid $F^{1/2,1/2}_{\boldsymbol{p}_8}$, dotted $G_{1/2,1/2,\boldsymbol{p}_8}$, and dashed $G^{1/2,1/2}_{\boldsymbol{p}_8}$

that $G_{1/2,1/2,\boldsymbol{p}_8}(\cdot) \equiv G_{1/2,1/2,3/4}(\cdot)$ by (2.16).

In general we see that, extending greatly the sums from [10] to the linear combinations considered here, already the primary semistable merging approximations appear to be reasonably good, while the corresponding asymptotic expansions may be working incredibly well in a variety of different circumstances even for small n.

2.5. Proofs

The proof of Theorem 2.1 is based on Esseen's classical result (Theorem 5.2 in [35]), which we record here in a special case closest to our application.

Lemma 2.1. *Let F be a distribution function and G be a function of bounded variation on $\mathbb{R}$ with Fourier – Stieltjes transforms $\mathbf{f}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \mathrm{d}F(x)$ and $\mathbf{g}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \mathrm{d}G(x)$, $t \in \mathbb{R}$, such that $G(-\infty) = \lim_{x\to-\infty} G(x) = 0 = F(-\infty)$ and the derivative G' of G exists and is bounded on the whole $\mathbb{R}$. Then*

$$\sup_{x\in\mathbb{R}} |F(x) - G(x)| \leq \frac{b}{2\pi} \int_{-T}^{T} \left| \frac{\mathbf{f}(t) - \mathbf{g}(t)}{t} \right| \mathrm{d}t + c_b \frac{\sup_{x\in\mathbb{R}} |G'(x)|}{T}$$

for every choice of $T > 0$ and $b > 1$, where $c_b > 0$ is a constant depending only on b, which can be given as $c_b = 4bd_b^2/\pi$, where $d_b > 0$ is the unique root d of the equation $\frac{4}{\pi} \int_0^d \frac{\sin^2 u}{u^2} \mathrm{d}u = 1 + \frac{1}{b}$.

For $j = 1, 2, 5, 6$ the constants $C_j(\alpha, p)$ below are the same as in [10] and agree with the respective constants $c_j(\alpha, p)$ in [7], while the constants $C_j(\alpha, p)$ numbered numbered with $j = 7, 8, 9$ are the same as in [10]. The following lemma is Lemma 3 in [10], the proof of the first inequality is already in [7].

Lemma 2.2. *Uniformly in $\gamma \in (q, 1]$,*

$$\mathfrak{Re}\, y_\gamma^{\alpha,p}(t) \leq -C_1 |t|^\alpha, \quad t \in \mathbb{R}, \quad \textit{where} \quad C_1 = C_1(\alpha, p) = \left(\frac{2}{\pi}\right)^\alpha \frac{pq^{(2-\alpha)/\alpha}}{q - q^{2/\alpha}},$$

and

$$\left| y_\gamma^{\alpha,p}(t) \right| \leq v_{\alpha,p}(|t|), \quad t \in \mathbb{R},$$

where

$$v_{\alpha,p}(s) = \begin{cases} C_7\, s^\alpha, & \textit{if } \alpha \neq 1, \\ \left(C_7 + \frac{2p}{q} |\log_r s|\right) s, & \textit{if } \alpha = 1, \end{cases}$$

for every $s \geq 0$, and for the constant $C_7 = C_7(\alpha, p) > 0$ defined as

$$C_7(\alpha,p) = \begin{cases} \frac{2^{1-\alpha}}{q} + \frac{2^{1-\alpha}p}{q-q^{1/\alpha}}, & \text{if } \alpha < 1, \\ \frac{\max\{6\,,\,5+9p-8p\,\log_r 2\}}{2q}, & \text{if } \alpha = 1, \\ \frac{8p}{4^\alpha}\Big\{\frac{1}{q-q^{2/\alpha}} + \frac{1}{q-q^{(2\alpha-1)/\alpha}}\Big\}, & \text{if } \alpha > 1. \end{cases}$$

Proof of Theorem 2.1. The first step is to prove that the derivatives $(G^{\alpha,p}_{\boldsymbol{p}_n}(\cdot))'$ exist and are uniformly bounded in the strategies. In fact, first we claim that $I^{\alpha,p}_{j,\boldsymbol{p}_n} := \int_{-\infty}^{\infty} |t|^j \, |\mathbf{g}^{\alpha,p}_{\boldsymbol{p}_n}(t)| \, \mathrm{d}t < \infty$ for any $j \in \{0,1,2,\ldots\}$, which, referring to (2.15), implies that $G^{\alpha,p}_{\boldsymbol{p}_n}(\cdot)$ is arbitrary many times differentiable on $\mathbb{R}$. First note that by (2.16), Lemma 2.2 implies that for the characteristic function

$$\big|\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t)\big| = \exp\left\{\sum_{k=1}^{n} p_{k,n}\, \mathfrak{Re}\, y^{\alpha,p}_{\gamma_{k,n}}(t)\right\} \leq \mathrm{e}^{-C_1|t|^\alpha}, \quad t \in \mathbb{R}.$$

Proceeding for $\alpha = 1$, for which $I_j := \int_{-\infty}^{\infty} |t|^j \, \mathrm{e}^{-C_1|t|} \mathrm{d}t < \infty$, using (2.17), Lemma 2.2 and the triviality $\overline{p}_n \leq 1$, we obtain

$$\begin{aligned} I^{1,p}_{j,\boldsymbol{p}_n} \leq I_j + \frac{C_7^2 I_{j+2}}{2} + \frac{4C_7 p}{q} \sum_{k=1}^{n} p_{k,n} \int_0^\infty |t|^{j+1}\, \mathrm{e}^{-C_1|t|} \big|\log_r(p_{k,n}|t|)\big|(p_{k,n}|t|)\,\mathrm{d}t \\ + \frac{4p^2}{q^2} \sum_{k=1}^{n} p_{k,n} \int_0^\infty |t|^{j+1}\, \mathrm{e}^{-C_1|t|} \Big\{\log_r(p_{k,n}|t|)\sqrt{p_{k,n}|t|}\Big\}^2 \mathrm{d}t + \frac{I_{j+2}}{2q}. \end{aligned}$$

Breaking the k-th integral under both sums at $1/p_{k,n}$, using that $|\log_r s| s \leq l_p := (\log_r \mathrm{e})/\mathrm{e}$, $|\log_r s|\sqrt{s} \leq 2l_p$, $s \in (0,1)$, and $\log_r x \leq c_p x$, $x \geq 1$, for $c_p = 1/(\mathrm{e} \log r)$, where $\log = \log_{\mathrm{e}}$, and then extending all resulting integrals to $(0,\infty)$ again, we get

$$I^{1,p}_{j,\boldsymbol{p}_n} \leq I_j + \frac{C_7^2 I_{j+2}}{2} + \frac{2C_7 p}{q}\big[l_p I_{j+1} + c_p I_{j+2}\big] + \frac{2p^2}{q^2}\big[4l_p^2 I_{j+1} + c_p^2 I_{j+4}\big] + \frac{I_{j+2}}{2q} =: M^{1,p}_j$$

for all $\boldsymbol{p}_n$. The argument is similar for $\alpha \neq 1$; in fact it is given below for $j = 0$.

Thus, writing (2.15) for the first derivative and using the usual Fourier inversion formula, (2.17) and Lemma 2.2 again, for $\alpha \neq 1$ we obtain

$$
\begin{aligned}
\left|\left(G_{\boldsymbol{p}_n}^{\alpha,p}(x)\right)'\right| &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx}\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)\mathrm{d}t\right| \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-C_1|t|^\alpha}\Bigg[1+\frac{1}{2}\sum_{k=1}^{n} v_{\alpha,p}^2(|t|p_{k,n}^{1/\alpha}) \\
&\quad +|ts_1^{\alpha,p}|\sum_{k=1}^{n} p_{k,n}^{1/\alpha} v_{\alpha,p}(|t|p_{k,n}^{1/\alpha}) + \frac{t^2}{2}\left\{(s_1^{\alpha,p})^2+\frac{p}{q-q^{2/\alpha}}\right\}\sum_{k=1}^{n} p_{k,n}^{2/\alpha}\Bigg]\mathrm{d}t \\
&\leq \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-C_1 t^\alpha}\Bigg[1+\frac{C_7^2\overline{p}_n t^{2\alpha}}{2}+C_7|s_1^{\alpha,p}|\overline{p}_n^{1/\alpha}\, t^{1+\alpha} \\
&\quad +\frac{t^2}{2}\left\{(s_1^{\alpha,p})^2+\frac{p}{q-q^{2/\alpha}}\right\}\overline{p}_n^{(2-\alpha)/\alpha}\Bigg]\mathrm{d}t \leq M^{\alpha,p},
\end{aligned}
$$

where the constant $M^{\alpha,p}$ is obtained upon replacing $\overline{p}_n$ by 1, and where we used the trivial inequality $\sum_{k=1}^n p_{k,n}^\beta \leq \overline{p}_n^{\beta-1}$, $\beta > 1$. For $\alpha = 1$, the proof is done above, so that the bound $M^{1,p}$ on the first derivative can be taken as $M_0^{1,p}$ above.

Now we turn to the proof of the theorem, which is an extension of the proof of the Proposition in [10]; whenever possible, we use the same or analogous notation as there. We may skip some detail for $\alpha \neq 1$.

Using Esseen's inequality, we get

$$
\begin{aligned}
\Delta_{\boldsymbol{p}_n}^{\alpha,p} &:= \sup_{x\in\mathbb{R}}\left|\mathbf{P}\{S_{\boldsymbol{p}_n}^{\alpha,p}\leq x\}-G_{\boldsymbol{p}_n}^{\alpha,p}(x)\right| \\
&\leq \frac{b}{2\pi}\int_{-T_n^{\alpha,p}}^{T_n^{\alpha,p}} \frac{\left|\mathbf{E}(\mathrm{e}^{\mathrm{i}tS_{\boldsymbol{p}_n}^{\alpha,p}})-\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)\right|}{|t|}\,\mathrm{d}t + c_b\frac{M_{\alpha,p}}{T_n^{\alpha,p}} =: \frac{b}{2\pi}\Delta_{\boldsymbol{p}_n,1}^{\alpha,p}+c_b\,\Delta_{\boldsymbol{p}_n,2}^{\alpha,p}\,,
\end{aligned}
$$

where $T_n^{\alpha,p} = 2K^{1/\alpha}/\overline{p}_n^{1/\alpha}$, and on the constant $K = K_{\alpha,p} > 0$ we will introduce some restrictions as we go along. By the choice of $T_n^{\alpha,p}$ we have $\Delta_{\boldsymbol{p}_n,2}^{\alpha,p} = O(\overline{p}_n^{1/\alpha})$. The estimation of the other term requires some further notation. The characteristic functions of $S_{\boldsymbol{p}_n}^{\alpha,p}$ and $W_{\boldsymbol{p}_n}^{\alpha,p}$ can be written in the form

$$
\mathbf{E}\left(\mathrm{e}^{\mathrm{i}tS_{\boldsymbol{p}_n}^{\alpha,p}}\right) = \mathrm{e}^{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n)}\prod_{k=1}^{n}\mathbf{E}\left(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha}X_k}\right) = \mathrm{e}^{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n)}\prod_{k=1}^{n}\left(1+y_{k,n}^{\alpha,p}(t)\right)
$$

and

$$
\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t) = \mathbf{E}\left(\mathrm{e}^{\mathrm{i}tW_{\boldsymbol{p}_n}^{\alpha,p}}\right) = \mathrm{e}^{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n)}\prod_{k=1}^{n}\mathrm{e}^{z_{\alpha,p}(p_{k,n}^{1/\alpha}t)}, \quad t\in\mathbb{R},
$$

where $y_{k,n}^{\alpha,p}(t)=\mathbf{E}\big(\exp\big\{\mathrm{i}tp_{k,n}^{1/\alpha}X_k\big\}-1\big)$ and $z_{\alpha,p}(s)=y_1^{\alpha,p}(s)-\mathrm{i}s_1^{\alpha,p}s$, $s\in\mathbb{R}$. Notice that $z_{1,p}(s)=y_1^{1,p}(s)$. Continuing the transformations, we may write

$$\begin{aligned}
\mathbf{E}\Big(\mathrm{e}^{\mathrm{i}tS_{\boldsymbol{p}_n}^{\alpha,p}}\Big) &= \exp\left\{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n)+\sum_{k=1}^n\log(1+y_{k,n}^{\alpha,p}(t))\right\}\\
&= \exp\left\{-\mathrm{i}t\frac{p}{q}H_{\alpha,p}(\boldsymbol{p}_n)+\sum_{k=1}^n z_{\alpha,p}(p_{k,n}^{1/\alpha}t)+\sum_{k=1}^n R_{k,n,1}^{\alpha,p}(t)+\sum_{k=1}^n w_{k,n}^{\alpha,p}(t)\right\}\\
&= \mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t)\exp\left\{\sum_{k=1}^n\big(R_{k,n,1}^{\alpha,p}(t)+w_{k,n}^{\alpha,p}(t)\big)\right\}\\
&= \mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t)\left[1+\sum_{k=1}^n\big(R_{k,n,1}^{\alpha,p}(t)+w_{k,n}^{\alpha,p}(t)\big)+R_{n,2}^{\alpha,p}(t)\right]\\
&= \mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t)\left[1-\frac{1}{2}\sum_{k=1}^n(y_{k,n}^{\alpha,p}(t))^2+R_{n,1}^{\alpha,p}(t)+R_{n,2}^{\alpha,p}(t)+R_{n,3}^{\alpha,p}(t)\right],
\end{aligned}$$

where the error terms are $w_{k,n}^{\alpha,p}(t)=\log(1+y_{k,n}^{\alpha,p}(t))-y_{k,n}^{\alpha,p}(t)$ and

$$R_{n,1}^{\alpha,p}(t)=\sum_{k=1}^n R_{k,n,1}^{\alpha,p}(t)=\sum_{k=1}^n\Big(y_{k,n}^{\alpha,p}(t)-z_{\alpha,p}(p_{k,n}^{1/\alpha}t)\Big),$$

$$R_{n,2}^{\alpha,p}(t)=\sum_{l=2}^\infty\frac{1}{l!}\left[\sum_{k=1}^n w_{k,n}^{\alpha,p}(t)+R_{k,n,1}^{\alpha,p}(t)\right]^l,\quad R_{n,3}^{\alpha,p}(t)=\sum_{k=1}^n\sum_{l=3}^\infty(-1)^{l+1}\frac{1}{l}(y_{k,n}^{\alpha,p}(t))^l.$$

In general we use the simplifying convention $R_{n,j}^{\alpha,p}(t)=\sum_{k=1}^n R_{k,n,j}^{\alpha,p}(t)$, $j=1,3,6$. Finally, using the identity $y_{k,n}^{\alpha,p}(t)=y_1^{\alpha,p}(p_{k,n}^{1/\alpha}t)-\mathrm{i}tp_{k,n}^{1/\alpha}s_1^{\alpha,p}+R_{k,n,1}^{\alpha,p}(t)$, we obtain

$$\begin{aligned}
\mathbf{E}\Big(\mathrm{e}^{\mathrm{i}tS_{\boldsymbol{p}_n}^{\alpha,p}}\Big) &= \mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t)\Bigg[1-\frac{1}{2}\sum_{k=1}^n y_1^{\alpha,p}(p_{k,n}^{1/\alpha}t)^2+\mathrm{i}ts_1^{\alpha,p}\sum_{k=1}^n p_{k,n}^{1/\alpha}y_1^{\alpha,p}(p_{k,n}^{1/\alpha}t)\\
&\qquad+\frac{t^2}{2}\left\{(s_1^{\alpha,p})^2+\frac{p}{q-q^{2/\alpha}}\right\}\sum_{k=1}^n p_{k,n}^{2/\alpha}+R_{n,5}^{\alpha,p}(t)\Bigg]\\
&= \mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)+\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)R_{n,5}^{\alpha,p}(t)=\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)+R_{n,7}^{\alpha,p}(t),
\end{aligned}$$

where

$$R_{n,5}^{\alpha,p}(t)=\widetilde{R}_{n,1}^{\alpha,p}(t)+R_{n,2}^{\alpha,p}(t)+R_{n,3}^{\alpha,p}(t)+R_{n,6}^{\alpha,p}(t)=R_{n,4}^{\alpha,p}(t)+R_{n,6}^{\alpha,p}(t),$$

$$\widetilde{R}_{n,1}^{\alpha,p}(t)=\sum_{k=1}^n\widetilde{R}_{k,n,1}^{\alpha,p}(t)=\sum_{k=1}^n\left[R_{k,n,1}^{\alpha,p}(t)-p_{k,n}^{2/\alpha}\frac{t^2\,p}{2\,(q-q^{2/\alpha})}\right]$$

and

$$R_{n,6}^{\alpha,p}(t) = \sum_{k=1}^{n} R_{k,n,6}^{\alpha,p}(t) = \sum_{k=1}^{n} \left[-\frac{1}{2} R_{k,n,1}^{\alpha,p}(t)^2 - R_{k,n,1}^{\alpha,p}(t) \left\{ y_1^{\alpha,p}(tp_{k,n}^{1/\alpha}) - \mathrm{i}tp_{k,n}^{1/\alpha} s_1^{\alpha,p} \right\} \right].$$

Now we turn to the estimation of the remainder terms. By definition and (2.1),

$$\begin{aligned} y_{k,n}^{\alpha,p}(t) &= \mathbf{E}\Big(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} X_k} - 1\Big) = \int_0^\infty \Big(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} x} - 1\Big)\,\mathrm{d}F_{\alpha,p}(x) \\ &= \sum_{l=1}^{\infty} \Big(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}} - 1\Big) q^{l-1}\, p, \end{aligned}$$

and by (2.4),

$$z_{\alpha,p}(p_{k,n}^{1/\alpha} t) = \sum_{l=0}^{-\infty} \Big(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}} - 1 - \mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}\Big)\, q^{l-1} p + \sum_{l=1}^{\infty} \Big(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}} - 1\Big)\, q^{l-1} p.$$

Thus, using the inequality $|\mathrm{e}^{\mathrm{i}u} - 1 - \mathrm{i}u| \le u^2/2$, $u \in \mathbb{R}$ (Lemma 4 in [7]), we obtain

$$\begin{aligned} |R_{k,n,1}^{\alpha,p}(t)| &= \Big| y_{k,n}^{\alpha,p}(t) - z_{\alpha,p}(tp_{k,n}^{1/\alpha}) \Big| \le \sum_{l=0}^{-\infty} \Big| \Big(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}} - 1 - \mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}\Big)\, q^{l-1} p \Big| \\ &\le \frac{|t|^2 p_{k,n}^{2/\alpha} p}{2q} \sum_{l=0}^{\infty} q^{(\frac{2}{\alpha}-1)l} = \frac{|t|^2 p_{k,n}^{2/\alpha} p}{2q}\, \frac{1}{1 - q^{2\alpha - 1}} = |t|^2\, C_2\, p_{k,n}^{2/\alpha}, \end{aligned}$$

where $C_2 = C_2(\alpha, p) = p/(2(q - q^{2/\alpha}))$. Since

$$\widetilde{R}_{k,n,1}^{\alpha,p}(t) = -\sum_{l=0}^{-\infty} \left(\mathrm{e}^{\mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha}} - 1 - \mathrm{i}tp_{k,n}^{1/\alpha} r^{l/\alpha} - \frac{(\mathrm{i}t)^2 p_{k,n}^{2/\alpha} r^{2l/\alpha}}{2} \right) q^{l-1} p,$$

using this time the inequality $\left|\mathrm{e}^{\mathrm{i}u} - 1 - \mathrm{i}u - \frac{(\mathrm{i}u)^2}{2}\right| \le \frac{|u|^3}{6}$, $u \in \mathbb{R}$, we obtain

$$\big|\widetilde{R}_{k,n,1}^{\alpha,p}(t)\big| \le \frac{|t|^3 p_{k,n}^{3/\alpha} p}{6\,q} \sum_{l=0}^{\infty} q^{(\frac{3}{\alpha}-1)l} = \frac{|t|^3 p_{k,n}^{3/\alpha} p}{6\,q}\, \frac{1}{1 - q^{\frac{3}{\alpha}-1}} = |t|^3\, \widetilde{C}_2\, p_{k,n}^{3/\alpha},$$

where $\widetilde{C}_2(\alpha, p) = p/(6(q - q^{3/\alpha}))$. Summing these bounds for $k = 1, \dots, n$, we get

$$\text{(2.24)} \qquad \big|R_{n,1}^{\alpha,p}(t)\big| \le C_2 |t|^2 \sum_{k=1}^{n} p_{k,n}^{2/\alpha} \quad \text{and} \quad \big|\widetilde{R}_{n,1}^{\alpha,p}(t)\big| \le \widetilde{C}_2 |t|^3 \sum_{k=1}^{n} p_{k,n}^{3/\alpha}.$$

Introduce $x_{k,n}^{\alpha,p}(t) = y_{k,n}^{\alpha,p}(t)/p_{k,n}$. Then the calculation on pages 320–321 in [10], which goes back to page 837 in [7], now yields

$$|x_{k,n}^{\alpha,p}(t)| \le \begin{cases} C_5|t|^\alpha + C_6|t|p_{k,n}^{1/\alpha-1}, & \text{if } \alpha \neq 1, \\ |t|\left(r + \frac{p}{q}\log_r \frac{2}{|t|p_{k,n}}\right), & \text{if } \alpha = 1, \end{cases}$$

for $|t| \le 2q^{1/\alpha}/p_{k,n}^{1/\alpha}$, where

$$C_5 = C_5(\alpha,p) = 2^{1-\alpha}\left\{\frac{1}{q} + \frac{p}{q-q^{1/\alpha}}\right\} \quad \text{and} \quad C_6 = C_6(\alpha,p) = \frac{p}{q^{1/\alpha}-q}.$$

Notice that $C_5 > 0$ and $C_6 < 0$ for $\alpha < 1$, so that $|x_{k,n}^{\alpha,p}(t)| \le C_5|t|^\alpha$ for $\alpha < 1$. On the other hand, $C_6 > 0$, but C_5 can be both positive and negative for $\alpha > 1$. Therefore, we need the following argument. If $|t| \le T_n^{\alpha,p} = 2K^{1/\alpha}/\overline{p}_n^{1/\alpha}$ then $|C_5||t|^\alpha \le |t||C_5||T_n^{\alpha,p}|^{\alpha-1} \le |t|p_{k,n}^{(1-\alpha)/\alpha}\{2^{\alpha-1}K^{(\alpha-1)/\alpha}|C_5|\}$, where the expression in the last pair of curly braces is < 1 if K is small enough, in which case $|x_{k,n}^{\alpha,p}(t)| \le (C_6+1)|t|\,p_{k,n}^{(1-\alpha)/\alpha}$. An easy monotonicity argument on the upper bounds implies that if K is small enough, then there exists $L = L_{\alpha,p} \in (0,1)$ such that $|y_{k,n}^{\alpha,p}(t)| \le L < 1$ for t in the interval $[-T_n^{\alpha,p}, T_n^{\alpha,p}]$. Thus we have the estimates

$$|w_{k,n}^{\alpha,p}(t)| = |\log(1 + y_{k,n}^{\alpha,p}(t)) - y_{k,n}^{\alpha,p}(t)| \le C_8|y_{k,n}^{\alpha,p}(t)|^2,$$

and

$$|R_{k,n,3}^{\alpha,p}(t)| \le \sum_{l=3}^{\infty} \frac{1}{l}|y_{k,n}^{\alpha,p}(t)|^l \le C_9|y_{k,n}^{\alpha,p}(t)|^3$$

where, by the same elementary calculations as on page 323 in [10],

$$C_8 = C_8(\alpha,p) = \frac{1}{6} + \frac{1}{3}\frac{1}{1-L_{\alpha,p}} \quad \text{and} \quad C_9 = C_9(\alpha,p) = \frac{1}{12} + \frac{1}{4}\frac{1}{1-L_{\alpha,p}}\,.$$

Using these bounds, the second statement of Lemma 2.1 and (2.24), we get

$$\begin{aligned} |R_{n,5}^{\alpha,p}(t)| \le\ & \widetilde{C}_2|t|^3\sum_{k=1}^{n} p_{k,n}^{3/\alpha} + R_{n,8}^{\alpha,p}(t) + C_9\sum_{k=1}^{n}|y_{k,n}^{\alpha,p}(t)|^3 + \frac{C_2^2|t|^4}{2}\sum_{k=1}^{n}p_{k,n}^{4/\alpha} \\ & + C_2t^2\sum_{k=1}^{n}p_{k,n}^{2/\alpha}v_{\alpha,p}(tp_{k,n}^{1/\alpha}) + C_2|s_1^{\alpha,p}||t|^3\sum_{k=1}^{n}p_{k,n}^{3/\alpha}, \end{aligned} \tag{2.25}$$

for all $t \in [-T_n^{\alpha,p}, T_n^{\alpha,p}]$, where $R_{n,8}^{\alpha,p}(t)$ is an upper bound on $|R_{n,2}^{\alpha,p}(t)|$, given by

$$R_{n,8}^{\alpha,p}(t) = \sum_{l=2}^{\infty}\frac{1}{l!}\left[C_8\sum_{k=1}^{n}|y_{k,n}^{\alpha,p}(t)|^2 + C_2|t|^2\sum_{k=1}^{n}p_{k,n}^{2/\alpha}\right]^l.$$

For simplicity we now separate the three main cases: $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$. In the followings we will use the simple identity $\int_0^\infty t^\eta \mathrm{e}^{-ct^\alpha} \mathrm{d}t = \frac{\Gamma((\eta+1)/\alpha)}{\alpha c^{(\eta+1)/\alpha}}$, for $\eta > -1$, $c > 0$ and the inequality $\sum_{k=1}^n p_{k,n}^\beta \leq \overline{p}_n^{\beta-1}$ for $\beta > 1$.

Consider first the case $\alpha \in (0,1)$. Since $|y_{k,n}^{\alpha,p}(t)| = |p_{k,n} x_{k,n}^{\alpha,p}(t)| \leq p_{k,n} C_5 |t|^\alpha$ and $v_{\alpha,p}(|t|) = C_7 |t|^\alpha$, we have by (2.25),

$$\begin{aligned}\Delta_{\boldsymbol{p}_n,1}^{\alpha,p} \leq{}& \overline{p}_n^{\frac{3}{\alpha}-1} 2\widetilde{C}_2 \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} + \overline{p}_n^2 2 C_5^3 C_9 \frac{\Gamma(2.3)}{\alpha C_1^3} + \overline{p}_n^{\frac{4}{\alpha}-1} C_2^2 \frac{\Gamma(4/\alpha)}{\alpha C_1^{4/\alpha}} \\ &+ \overline{p}_n^{\frac{2}{\alpha}} 2 C_2 C_7 \frac{\Gamma((2+\alpha)/\alpha)}{\alpha C_1^{(2+\alpha)/\alpha}} + \overline{p}_n^{\frac{3}{\alpha}-1} 2 C_2 \left| s_1^{\alpha,p} \right| \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} \\ &+ 2 \int_0^{T_n^{\alpha,p}} \frac{1}{t} \mathrm{e}^{-C_1 t^\alpha} |R_{n,8}^{\alpha,p}(t)| \, \mathrm{d}t.\end{aligned}$$

Substituting the bounds into $R_{n,8}^{\alpha,p}(t)$, we obtain

$$\begin{aligned}|R_{n,8}^{\alpha,p}(t)| &\leq \sum_{l=2}^\infty \frac{1}{l!} \left[C_8 C_5^2 |t|^{2\alpha} \overline{p}_n + C_2 |t|^2 \overline{p}_n^{(2-\alpha)/\alpha} \right]^l \\ &\leq \sum_{l=2}^\infty \frac{1}{l!} \left[\left(C_8 C_5^2 + C_2 |t|^{2-2\alpha} \overline{p}_n^{(2-2\alpha)/\alpha} \right) |t|^{2\alpha} \overline{p}_n \right]^l \\ &\leq \sum_{l=2}^\infty \frac{1}{l!} \left[\left(C_8 C_5^2 + C_2 (2K^{1/\alpha})^{2-2\alpha} \right) |t|^{2\alpha} \overline{p}_n \right]^l,\end{aligned}$$

where we used that $|t|^{2-2\alpha} \overline{p}_n^{2/\alpha-2} \leq (T_n^{\alpha,p})^{2-2\alpha} \overline{p}_n^{2/\alpha-2} = (2K^{1/\alpha})^{2-2\alpha}$. Then the same calculation as in [10], page 325, yields

$$2 \int_0^{T_n^{\alpha,p}} \frac{\mathrm{e}^{-C_1 t^\alpha}}{t} |R_{n,8}^{\alpha,p}(t)| \, \mathrm{d}t \leq \overline{p}_n^2 \frac{2}{\alpha 4^\alpha C_1^2 K^2} \frac{3R^2 - 2R^3}{(1-R)^2},$$

provided that $K = K_{\alpha,p}$ is small enough to make

$$R = R(\alpha,p) = 2^\alpha K \frac{C_8 C_5^2 + C_2 (2K^{1/\alpha})^{2-2\alpha}}{C_1} < 1.$$

After an easy check on the powers of $\overline{p}_n$ the proof is ready in this case.

Now consider the case $\alpha = 1$. Elementary analysis shows that for each $\delta \in (0,1)$ the function $f(t) = t^\delta \left(r + \frac{p}{q} \log_r \frac{2}{\overline{p}_n t} \right)$ is monotone increasing on $(0, T_n^{1,p})$ if $K < \mathrm{e}^{-1/\delta}$. Recall that $T_n^{1,p} = 2K/\overline{p}_n$. The monotonicity of f easily implies that

$$(p_{k,n} t)^\delta \left(r + \frac{p}{q} \log_r \frac{2}{p_{k,n} t} \right) \leq (2K)^\delta \left(r + \frac{p}{q} \log_r \frac{1}{K} \right)$$

for $t \in (0, T_n^{1,p})$, $k = 1, 2, \ldots, n$. Applying this for $\delta = 1/3$ we get

$$\frac{y_{k,n}^{1,p}(t)^3}{t} = p_{k,n}^3 t^2 \left(r + \frac{p}{q} \log_r \frac{2}{t p_{k,n}} \right)^3 \leq p_{k,n}^2 t \left[\left(r + \frac{p}{q} \log_r \frac{1}{K} \right) (2K)^{1/3} \right]^3,$$

if $K < \mathrm{e}^{-3}$. Using also the inequality $t p_{k,n} \log_r \frac{1}{t p_{k,n}} \leq 2K \log_r \frac{1}{2K}$ and integrating the bounds in (2.25), we obtain

$$\begin{aligned}\Delta_{\boldsymbol{p}_n,1}^{1,p} \leq \overline{p}_n^2 2\widetilde{C}_2 \frac{\Gamma(3)}{C_1^3} + \overline{p}_n 2C_9 \left[\left(r + \frac{p}{q} \log_r \frac{1}{K} \right) (2K)^{1/3} \right]^3 \frac{\Gamma(2)}{C_1^2} \\ + \overline{p}_n^3 C_2^2 \frac{\Gamma(4)}{C_1^4} + \overline{p}_n^2 2C_2 C_7 \frac{\Gamma(3)}{C_1^3} + \overline{p}_n \frac{4C_2 p \Gamma(2)}{q C_1^2} 2K \log_r \frac{1}{2K} + \Delta_{\boldsymbol{p}_n,3}^{1,p},\end{aligned}$$

where $\Delta_{\boldsymbol{p}_n,3}^{1,p} = 2\int_0^{T_n^{1,p}} \mathrm{e}^{-C_1 t}\, t^{-1}\, R_{n,8}^{1,p}(t)\, \mathrm{d}t$. The monotonicity of f also implies the inequality $p_{k,n}(r + \frac{p}{q} \log_r \frac{2}{t p_{k,n}})^2 \leq \overline{p}_n (r + \frac{p}{q} \log_r \frac{2}{t \overline{p}_n})^2$, $k = 1, 2, \ldots, n$, if $K < \mathrm{e}^{-2}$. Hence we obtain

$$\begin{aligned} R_{n,8}^{1,p}(t) &\leq \sum_{l=2}^{\infty} \frac{1}{l!} \left[C_2 t^2 \overline{p}_n + C_8 t^2 \sum_{k=1}^{n} p_{k,n}^2 \left(r + \frac{p}{q} \log_r \frac{2}{t p_{k,n}} \right)^2 \right]^l \\ &\leq \sum_{l=2}^{\infty} \frac{1}{l!} \left[C_2 t^2 \overline{p}_n + C_8 t^2 \overline{p}_n \left(r + \frac{p}{q} \log_r \frac{2}{t \overline{p}_n} \right)^2 \right]^l, \end{aligned}$$

and since $1 - \frac{2}{l} + \frac{2\delta}{l} \geq \delta$ for every $l \geq 2$, the inequality

$$\left[C_2 + C_8 \left(r + \frac{p}{q} \log_r \frac{2}{\overline{p}_n t} \right)^2 \right] t^{1 - \frac{2}{l} + \frac{2\delta}{l}} \leq \left[C_2 + C_8 \left(r + \frac{p}{q} \log_r \frac{1}{K} \right)^2 \right] (T_n^{1,p})^{1 - \frac{2}{l} + \frac{2\delta}{l}}$$

holds on $(0, T_n^{1,p})$, if K is so small that $K < \mathrm{e}^{-2/\delta}$. Substituting these bounds into $\Delta^{1,p}_{\boldsymbol{p}_n,3}$ and using that $C_1(1,p) = 2/\pi$, we get

$$\Delta^{1,p}_{\boldsymbol{p}_n,3} = \sum_{l=2}^{\infty} \frac{2\overline{p}_n^l}{l!} \int_0^{\frac{2K}{\overline{p}_n}} \left[C_2 t^2 + C_8 t^2 \left(r + \frac{p}{q} \log_r \frac{2}{t\overline{p}_n} \right)^2 \right]^l \frac{\mathrm{e}^{-\frac{2}{\pi}t}}{t} \,\mathrm{d}t$$

$$= \sum_{l=2}^{\infty} \frac{2\overline{p}_n^l}{l!} \int_0^{\frac{2K}{\overline{p}_n}} \left[\left\{ C_2 + C_8 \left(r + \frac{p}{q} \log_r \frac{2}{t\overline{p}_n} \right)^2 \right\} t^{1-\frac{2}{l}+\frac{2\delta}{l}} \right]^l t^{l+1-2\delta}\, \mathrm{e}^{-\frac{2}{\pi}t} \,\mathrm{d}t$$

$$\le \frac{2\overline{p}_n^{2-2\delta}}{(2K)^{2-2\delta}} \sum_{l=2}^{\infty} \frac{(2K)^l}{l!} \left[C_2 + C_8 \left(r + \frac{p}{q} \log_r \frac{1}{K} \right)^2 \right]^l \left(\frac{\pi}{2}\right)^{l+2-2\delta} \Gamma(l+2-2\delta)$$

$$\le \frac{\overline{p}_n^{2-2\delta} \pi^{2-2\delta}}{2^{3-4\delta} K^{2-2\delta}} \sum_{l=2}^{\infty} (l+1) \left[\pi C_2 K + \pi C_8 \left(r + \frac{p}{q} \log_r \frac{1}{K} \right)^2 K \right]^l$$

$$= \frac{\overline{p}_n^{2-2\delta} \pi^{2-2\delta}}{2^{3-4\delta} K^{2-2\delta}} \sum_{l=2}^{\infty} (l+1) R^l = \overline{p}_n^{2-2\delta} \frac{\pi^{2-2\delta}(3R^2 - 2R^3)}{2^{3-4\delta} K^{2-2\delta} (1-R)^2},$$

provided that K is small enough to make

$$R = R_{1,p} = \pi C_8(1,p) \left(\frac{1}{q} + \frac{p}{q} \log_r \frac{1}{K_{1,p}} \right)^2 K_{1,p} + \pi C_2(1,p) K_{1,p} < 1.$$

For simplicity here we used the inequality $\Gamma(l+2-2\delta) < \Gamma(l+2) = (l+1)!$ for all $l = 2, 3, \ldots$. Choosing now $\delta < 1/2$ and collecting all terms, we see that the order is indeed $O(\overline{p}_n)$ as claimed.

In the final case $\alpha > 1$, we have $|y^{\alpha,p}_{k,n}(t)| = |p_{k,n} x^{\alpha,p}_{k,n}(t)| \le (C_6+1)|t| p_{k,n}^{1/\alpha}$ and $v_{\alpha,p}(|t|) = C_7 |t|^\alpha$. Substituting into $\Delta^{\alpha,p}_{\boldsymbol{p}_n,1}$, by (2.25) we obtain

$$\begin{aligned}\Delta^{\alpha,p}_{\boldsymbol{p}_n,1} \le{}& \overline{p}_n^{\frac{3}{\alpha}-1} 2\widetilde{C}_2 \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} + \overline{p}_n^{\frac{3}{\alpha}-1} 2C_9 (C_6+1)^3 \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} + \overline{p}_n^{\frac{4}{\alpha}-1} C_2^2 \frac{\Gamma(4/\alpha)}{\alpha C_1^{4/\alpha}} \\ &+ \overline{p}_n^{\frac{2}{\alpha}} 2C_2 C_7 \frac{\Gamma((2+\alpha)/\alpha)}{\alpha C_1^{(2+\alpha)/\alpha}} + \overline{p}_n^{\frac{3}{\alpha}-1} 2C_2 \left| s_1^{\alpha,p} \right| \frac{\Gamma(3/\alpha)}{\alpha C_1^{3/\alpha}} \\ &+ 2 \int_0^{T_n^{\alpha,p}} \frac{\mathrm{e}^{-C_1 t^\alpha}}{t} \left| R^{\alpha,p}_{n,8}(t) \right| \mathrm{d}t.\end{aligned}$$

Using the inequality

$$|R^{\alpha,p}_{n,8}(t)| \le \sum_{l=2}^{\infty} \frac{1}{l!} \left[C_8 (C_6+1)^2 |t|^2 \overline{p}_n^{(2-\alpha)/\alpha} + C_2 |t|^2 \overline{p}_n^{(2-\alpha)/\alpha} \right]^l,$$

and referring again to [10], page 329, we get

$$\int_0^{T_n^{\alpha,p}} \frac{\mathrm{e}^{-C_1 t^\alpha}}{t} |R^{\alpha,p}_{n,8}(t)| \,\mathrm{d}t \le \begin{cases} O\big(\overline{p}_n\big), & \text{if } 1 < \alpha < 4/3, \\ O\big(\overline{p}_n^{2(2-\alpha)/\alpha}\big), & \text{if } 4/3 \le \alpha < 2. \end{cases}$$

Collecting all the terms and taking into account that $1/\alpha < (4-2\alpha)/\alpha$ if and only if $\alpha < 3/2$, the statement in the final case also follows. ■

Proof of Corollary 2.1. For simplicity we show for $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$ that

$$\sup_{x\in\mathbb{R}} \big|\mathbf{P}\big\{S_{\boldsymbol{p}_n}^{\alpha,p} \le x\big\} - G_{\alpha,p,\boldsymbol{p}_n}(x)\big| \le (1+\varepsilon)\frac{C_7^2}{2\pi\alpha C_1^2}\,\overline{p}_n,$$

$$\sup_{x\in\mathbb{R}} \big|\mathbf{P}\big\{S_{\boldsymbol{p}_n}^{1,p} \le x\big\} - G_{1,p,\boldsymbol{p}_n}(x)\big| \le (1+\varepsilon)\frac{p^2}{2q^2\pi C_1^2}\,\overline{p}_n \log_r^2 \frac{1}{\overline{p}_n},$$

and

$$\sup_{x\in\mathbb{R}} \big|\mathbf{P}\big\{S_{\boldsymbol{p}_n}^{\alpha,p} \le x\big\} - G_{\alpha,p,\boldsymbol{p}_n}(x)\big| \le (1+\varepsilon)\frac{\Gamma(2/\alpha)\left([s_1^{\alpha,p}]^2 + p/(q-q^{2/\alpha})\right)}{2\pi\alpha C_1^{2/\alpha}}\,\overline{p}_n^{\frac{2-\alpha}{\alpha}},$$

respectively, for all n large enough, where the strategy $\boldsymbol{p}_n$, with $\overline{p}_n \to 0$, corresponds to the given strategy $\boldsymbol{q}_n$ as described before Theorem 2.2. Then Corollary 2.1 follows by these statements exactly as Theorem 2.2 follows from Theorem 2.1.

First, if $\alpha < 1$, then $\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t)\sum_{k=1}^n p_{k,n}^2[y_{\gamma_{k,n}}^{\alpha,p}(t)]^2/2$ is the leading remainder term in $\mathbf{g}_{\boldsymbol{p}_n}^{\alpha,p}(t)$. We can estimate its inverse Fourier – Stieltjes transform $M_{\alpha,p,\boldsymbol{p}_n}^{(0,2)}(\cdot)$, which is not $G_{\alpha,p,\boldsymbol{p}_n}^{(0,2)}(\cdot)$, by the extended Gil-Pelaez – Rosén formula in Section 3:

$$M_{\alpha,p,\boldsymbol{p}_n}^{(0,2)}(x) = -\frac{1}{\pi}\int_0^\infty \frac{\mathfrak{Im}\big\{\mathrm{e}^{-\mathrm{i}tx}\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t)\frac{1}{2}\sum_{k=1}^n p_{k,n}^2[y_{\gamma_{k,n}}^{\alpha,p}(t)]^2\big\}}{t}\,\mathrm{d}t, \quad x\in\mathbb{R}.$$

Whence by (2.16) and Lemma 2.2,

$$\begin{aligned}\big|M_{\alpha,p,\boldsymbol{p}_n}^{(0,2)}(x)\big| &\le \frac{1}{2\pi}\int_0^\infty \frac{1}{t}\,\mathrm{e}^{\sum_{k=1}^n p_{k,n}\,\mathfrak{Re}\,y_{\gamma_{k,n}}^{\alpha,p}(t)}\sum_{k=1}^n p_{k,n}^2|y_{\gamma_{k,n}}^{\alpha,p}(t)|^2\mathrm{d}t\\ &\le \frac{C_7^2}{2\pi}\sum_{k=1}^n p_{k,n}^2\int_0^\infty \mathrm{e}^{-C_1t^\alpha}\,t^{2\alpha-1}\,\mathrm{d}t \le \frac{C_7^2}{2\pi\alpha C_1^2}\,\overline{p}_n\end{aligned}$$

for every $x\in\mathbb{R}$, finishing the first case.

Next, if $\alpha = 1$, then $\mathbf{g}_{1,p,\boldsymbol{p}_n}(t)\,\frac{p^2t^2}{2q^2}\sum_{k=1}^n p_{k,n}^2\log_r^2\frac{1}{p_{k,n}}$ is the leading remainder term in $\mathbf{g}_{\boldsymbol{p}_n}^{1,p}(t)$. For its inverse Fourier – Stieltjes transform $M_{1,p,\boldsymbol{p}_n}^{(2,0)}(\cdot)$, which differs from $G_{1,p,\boldsymbol{p}_n}^{(2,0)}(\cdot)$ only in a constant factor, we obtain

$$M_{1,p,\boldsymbol{p}_n}^{(2,0)}(x) = -\frac{1}{\pi}\int_0^\infty \frac{\mathfrak{Im}\big\{\mathrm{e}^{-\mathrm{i}tx}\mathbf{g}_{1,p,\boldsymbol{p}_n}(t)\frac{t^2p^2}{2q^2}\sum_{k=1}^n p_{k,n}^2\log_r^2\frac{1}{p_{k,n}}\big\}}{t}\,\mathrm{d}t, \quad x\in\mathbb{R},$$

by the extended Gil-Pelaez – Rosén formula. Thus, again by (2.16) and Lemma 2.2,

$$\begin{aligned}\left|M^{(2,0)}_{1,p,\boldsymbol{p}_n}(x)\right| &\leq \frac{p^2}{2q^2\pi}\int_0^\infty t\,\mathrm{e}^{\sum_{k=1}^n p_{k,n}\,\mathfrak{Re}\,y^{1,p}_{\gamma_{k,n}}(t)}\sum_{k=1}^n p_{k,n}^2\log_r^2\frac{1}{p_{k,n}}\,\mathrm{d}t\\ &\leq \frac{p^2}{2q^2\pi}\sum_{k=1}^n p_{k,n}^2\log_r^2\frac{1}{p_{k,n}}\int_0^\infty \mathrm{e}^{-C_1 t}\,t\,\mathrm{d}t \leq \frac{p^2}{2\pi q^2 C_1^2}\,\overline{p}_n\log_r^2\frac{1}{\overline{p}_n}\end{aligned}$$

for every $x\in\mathbb{R}$, where the last inequality comes from the fact that the function $x\mapsto x\log_r^2 x$ is monotone increasing near 0.

Finally, if $\alpha>1$, then the leading remainder term in $\mathbf{g}^{\alpha,p}_{\boldsymbol{p}_n}(t)$ is

$$\mathbf{m}^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(t)=\mathbf{g}_{\alpha,p,\boldsymbol{p}_n}(t)\frac{t^2}{2}\left\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\right\}\sum_{k=1}^n p_{k,n}^{2/\alpha}.$$

For its inverse Fourier – Stieltjes transform $M^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(\cdot)$, differing again from $G^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(\cdot)$ in a constant factor, by a final application of the extended Gil-Pelaez – Rosén formula we have

$$M^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(x)=-\frac{1}{\pi}\int_0^\infty\frac{\mathfrak{Im}\left\{\mathrm{e}^{-\mathrm{i}tx}\mathbf{m}^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(t)\right\}}{t}\,\mathrm{d}t,\quad x\in\mathbb{R}.$$

Therefore, using (2.16) and Lemma 2.2 for the last time, for all $x\in\mathbb{R}$ we obtain

$$\begin{aligned}\left|M^{(2,0)}_{\alpha,p,\boldsymbol{p}_n}(x)\right| &\leq \frac{1}{2\pi}\left\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\right\}\int_0^\infty t\mathrm{e}^{\sum_{k=1}^n p_{k,n}\,\mathfrak{Re}\,y^{\alpha,p}_{\gamma_{k,n}}(t)}\sum_{k=1}^n p_{k,n}^{2/\alpha}\,\mathrm{d}t\\ &\leq \frac{1}{2\pi}\left\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\right\}\sum_{k=1}^n p_{k,n}^{2/\alpha}\int_0^\infty \mathrm{e}^{-C_1t^\alpha}\,t\,\mathrm{d}t\\ &\leq \frac{\Gamma(2/\alpha)}{2\pi C_1^{2/\alpha}}\left\{\frac{p^2}{(q-q^{1/\alpha})^2}+\frac{p}{q-q^{2/\alpha}}\right\}\overline{p}_n^{(2-\alpha)/\alpha},\end{aligned}$$

completing the proof. ∎

Chapter 3.

Merging of linear combinations to semistable laws

3.1. Introduction

We need the definitions and the basic properties of semistable distributions and their domain of geometric partial attraction.

Let Y be an infinitely divisible real random variable with characteristic function $\phi(t) = \mathbf{E}(\mathrm{e}^{\mathrm{i}tY})$ in its Lévy form ([23], p. 70), given for each $t \in \mathbb{R}$ by

$$\phi(t) = \exp\left\{\mathrm{i}t\theta - \frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0} \beta_t(x)\,\mathrm{d}L(x) + \int_{0}^{\infty} \beta_t(x)\,\mathrm{d}R(x)\right\},$$

where

$$\beta_t(x) = \mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}$$

and where the constants $\theta \in \mathbb{R}$ and $\sigma \geq 0$ and the functions $L(\cdot)$ and $R(\cdot)$ are uniquely determined: $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and non-decreasing on $(0, \infty)$ with $R(\infty) = 0$, such that $\int_{-\varepsilon}^{0} x^2 \mathrm{d}L(x) + \int_{0}^{\varepsilon} x^2 \mathrm{d}R(x) < \infty$ for every $\varepsilon > 0$. We need a variant of this formula for $\phi(\cdot)$ in connection with a probabilistic representation of Y in [18]; the representation itself is not needed here. Let Ψ be the class of all non-positive, non-decreasing, right-continuous functions $\psi(\cdot)$, defined on $(0, \infty)$, such that $\int_{\varepsilon}^{\infty} \psi^2(s)\mathrm{d}s < \infty$ for each $\varepsilon > 0$. Then there is a one-to-one correspondence between the pairs of Lévy functions $L(\cdot)$ and $R(\cdot)$ and the pairs of functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ taken from Ψ if we put $\psi_1(s) = \inf\{x < 0 : L(x) > s\}$ and $\psi_2(s) = \inf\{x < 0 : -R(-x) > s\}$, $s > 0$, and, conversely, $L(x) = \inf\{s > 0 : \psi_1(s) \geq x\}$, $x < 0$, and $R(x) = -\inf\{s > 0 : \psi_2(s) \geq -x\}$, $x > 0$. Let $W(\psi_1, \psi_2, \sigma)$ be an infinitely divisible random variable with characteristic function

$$
\begin{aligned}
\mathbf{E}\Big(\mathrm{e}^{\mathrm{i}tW(\psi_1,\psi_2,\sigma)}\Big) &= \exp\bigg\{-\frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0}\beta_t(x)\,\mathrm{d}L(x) + \int_0^\infty \beta_t(x)\,\mathrm{d}R(x)\bigg\} \\
&= \exp\bigg\{-\frac{\sigma^2}{2}t^2 + \int_0^\infty \beta_t(\psi_1(u))\,\mathrm{d}u + \int_0^\infty \beta_t(-\psi_2(u))\,\mathrm{d}u\bigg\}, \qquad (3.1)
\end{aligned}
$$

where the second equality follows by Theorem 3 in [18]. The uniqueness of $\sigma, L(\cdot), R(\cdot)$ and the one-to-one correspondence immediately implies the uniqueness of the triple $\sigma, \psi_1(\cdot), \psi_2(\cdot)$. A concrete version of $W(\psi_1, \psi_2, \sigma)$ is given in [13] and, to keep complete accord with [13] as far as constants go, we also introduce $V(\psi_1, \psi_2, \sigma) = W(\psi_1, \psi_2, \sigma) + \theta(\psi_1) - \theta(\psi_2)$, where

$$
\theta(\psi) = \int_0^1 \frac{\psi(s)}{1+\psi^2(s)}\,\mathrm{d}s - \int_1^\infty \frac{\psi^3(s)}{1+\psi^2(s)}\,\mathrm{d}s, \quad \psi \in \Psi,
$$

and for its distribution function we put

$$
G_{\psi_1,\psi_2,\sigma}(x) = \mathbf{P}\big\{V(\psi_1,\psi_2,\sigma) \le x\big\}, \quad x \in \mathbb{R}. \qquad (3.2)
$$

Referring to [29], [24], [32] and [13] for background, we describe semistable laws in the present framework as follows: an infinitely divisible law $G_{\psi_1,\psi_2,\sigma}$ is semistable if and only if either $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ for some $\sigma > 0$, the normal distribution as a semistable distribution of exponent 2, or $(\psi_1, \psi_2, \sigma) = (\psi_1^\alpha, \psi_2^\alpha, 0)$, where

$$
\psi_j^\alpha(s) = -\frac{M_j(s)}{s^{1/\alpha}}, \quad s > 0,\ j = 1, 2, \qquad (3.3)
$$

for some $\alpha \in (0, 2)$, defining a semistable law of exponent α, where $M_1(\cdot)$ and $M_2(\cdot)$ are non-negative, right-continuous functions on $(0, \infty)$, either identically zero or bounded away from both zero and infinity, such that at least one of them is not identically zero, the functions $\psi_j^\alpha(\cdot)$ are non-decreasing and the multiplicative periodicity property $M_j(cs) = M_j(s)$ holds for all $s > 0$, for some constant $c > 1$, $j = 1, 2$. (The superscript α in ψ_j^α is a label, not a power exponent.) For the Lévy form this means that there exist non-negative bounded functions $M_L(\cdot)$ on $(-\infty, 0)$ and $M_R(\cdot)$ on $(0, \infty)$, one of which has strictly positive infimum and the other one either has strictly positive infimum or is identically zero, such that $L(x) = M_L(x)/|x|^\alpha$, $x < 0$, is left-continuous and non-decreasing on $(-\infty, 0)$ and $R(x) = -M_R(x)/x^\alpha$, $x > 0$, is right-continuous and non-decreasing on $(0, \infty)$ and $M_L(c^{1/\alpha}x) = M_L(x)$ for all $x < 0$ and $M_R(c^{1/\alpha}x) = M_R(x)$ for all $x > 0$, with the same period $c > 1$. Clearly, the two descriptions are equivalent.

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with the common distribution function $F(\cdot)$ and let $V(\psi_1, \psi_2, \sigma)$ and $G_{\psi_1,\psi_2,\sigma}$ be as in (3.2). Then F is in the domain of partial attraction of $G = G_{\psi_1,\psi_2,\sigma}$, written $F \in \mathbb{D}_{\mathrm{p}}(G)$, if for some centering and norming constants $c_{k_n} \in \mathbb{R}$ and $a_{k_n} > 0$ the convergence in distribution

$$\frac{1}{a_{k_n}}\left(\sum_{j=1}^{k_n} X_j - c_{k_n}\right) \xrightarrow{\mathcal{D}} V(\psi_1, \psi_2, \sigma), \tag{3.4}$$

holds along a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N} = \{1, 2, 3, \ldots\}$. The following theorem of Kruglov [29] highlights the importance of semistability; see [32] and [13] for further references. If (3.4) holds for some $F(\cdot)$ along some $\{k_n\}$ for which $\lim_{n\to\infty} k_{n+1}/k_n = c$ for some $c \in (1, \infty)$, then $G_{\psi_1,\psi_2,\sigma}$ is necessarily semistable and, when the exponent $\alpha < 2$, the common multiplicative period of $M_1(\cdot)$ and $M_2(\cdot)$ in (3.3) is the c from the latter growth condition on $\{k_n\}$. Conversely, for an arbitrary semistable distribution $G_{\psi_1,\psi_2,\sigma}$ there exists a distribution function $F(\cdot)$ for which (3.4) holds along some $\{k_n\} \subset \mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = c \quad \text{for some } c \in [1, \infty). \tag{3.5}$$

We say that a distribution $F(\cdot)$ is in the *domain of geometric partial attraction of G with rank $c \geq 1$*, written $F \in \mathbb{D}_{\mathrm{gp}}^{(c)}(G)$, if (3.4) holds along a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ satisfying (3.5). Clearly, if $\mathbb{D}_{\mathrm{gp}}(G) := \bigcup_{c\geq 1} \mathbb{D}_{\mathrm{gp}}^{(c)}(G) \neq \emptyset$ then G is semistable. Define $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) = \inf\{c > 1 : M_j(cs) = M_j(s),\ s > 0,\ j = 1, 2\}$, the minimal common period of the functions M_1, M_2 in ψ_1^α, ψ_2^α in (3.3), and $\mathbf{c}(G_{0,0,\sigma}) = 1$ for any $\sigma > 0$. Megyesi[3.14] showed that the entire domain $\mathbb{D}_{\mathrm{gp}}(G) = \bigcup_{c\geq 1} \mathbb{D}_{\mathrm{gp}}^{(c)}(G)$ of geometric partial attraction can be produced as $\mathbb{D}_{\mathrm{gp}}(G) = \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(G)$. Moreover, if $\mathbf{c}(G) = 1$ then the distribution G is necessarily stable.

The following characterization, that refines the one in [24], of $\mathbb{D}_{\mathrm{gp}}(G)$ is also taken from [32]. Fix a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ satisfying (3.5). If $c = 1$ then let $\gamma_x \equiv 1$, $x \geq 1$. If $c > 1$, then there exists an x_0 large enough such that for each $x > x_0$ there is a unique index $n^*(x)$ for which $k_{n^*(x)-1} < x \leq k_{n^*(x)}$. Then let $\gamma_x = x/k_{n^*(x)}$, for $x \in (x_0, \infty)$ and $\gamma_x = 1$ otherwise. We see by (3.5) that for any $\varepsilon > 0$ the inequality $c^{-1} - \varepsilon \leq \gamma_x \leq 1$ holds for all x large enough. We emphasize that γ_x depends on the subsequence $\{k_n\}_{n=1}^{\infty}$. For $s \in (0, 1)$ let $Q(s) = \inf\{x : F(x) \geq s\}$ be the quantile function of $F(\cdot)$, and let $Q_+(\cdot)$ denote its right-continuous version. Then (3.4) holds along the previously fixed subsequence $\{k_n\}_{n=1}^{\infty}$ for an arbitrary non-normal semistable distribution $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ if and only if

$$Q_+(s) = -s^{-1/\alpha}\, l(s)\big[M_1(1/\gamma_{1/s}) + h_1(s)\big] \;\text{ and}$$
$$(3.6)\qquad Q(1-s) = s^{-1/\alpha}\, l(s)\big[M_2(1/\gamma_{1/s}) + h_2(s)\big] \;\text{ for all }\; s \in (0,1),$$

where $l(\cdot)$ is a positive right-continuous function, slowly varying at zero, and the error terms $h_1(\cdot)$, $h_2(\cdot)$ are right-continuous functions such that $\lim_{s\downarrow 0} h_j(s) = 0$ if M_j is continuous, while if M_j has discontinuities then $h_j(s)$ may not go to zero but $\lim_{n\to\infty} h_j(t/k_n) = 0$ for $t \in C(M_j)$, $j = 1, 2$, where $C(f)$ stands for the set of continuity points of the function f. (The slightly different form of the quantile function here and in [32], p. 412, and [13] is due to the inverse relation between the two γ functions: instead of the $\gamma(\cdot)$ in [32] and [13], here we use $\gamma(s) = 1/\gamma_{1/s}$.) Conversely, if the $Q(\cdot)$ of $F(\cdot)$ satisfies (3.6), then $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ and

$$\frac{\sum_{j=1}^{k_n} X_j - k_n \int_{1/k_n}^{1-1/k_n} Q(u)\mathrm{d}u}{k_n^{1/\alpha}\, l(1/k_n)} \xrightarrow{\mathcal{D}} V(\psi_1^\alpha, \psi_2^\alpha, 0),$$

where $X_1, X_2, \ldots$ are independent with the common distribution function F.

The form (3.6) can be simplified for the simplest possible subsequence when (3.4) holds for $k_n \equiv \lfloor \mathbf{c}^n \rfloor$ for $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) > 1$. Then, as shown in [32],

$$Q_+(s) = -s^{-1/\alpha}\, l(s)\big[M_1(s) + h_1(s)\big] \;\text{ and}$$
$$(3.7)\qquad Q(1-s) = s^{-1/\alpha}\, l(s)\big[M_2(s) + h_2(s)\big] \;\text{ for all }\; s \in (0,1),$$

so we can just forget about the strange argument $1/\gamma_{1/s} = s\lfloor \mathbf{c}^{\lceil \log_{\mathbf{c}} \lceil 1/s \rceil \rceil} \rfloor$. Here $\lfloor y \rfloor = \max\{m \in \mathbb{Z} : m \le y\}$ and $\lceil y \rceil = \min\{m \in \mathbb{Z} : m \ge y\}$ denote the integer part and the ceiling of $y \in \mathbb{R}$ and $\log_{\mathbf{c}}$ stands for the logarithm to the base $\mathbf{c}$.

Let $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ be a fixed distribution function, where $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ is an arbitrary non-normal semistable distribution with characteristic exponent $\alpha \in (0, 2)$. Let $X_1, X_2, \ldots$ be independent random variables with the common distribution function $F(\cdot)$. Then $X_1, X_2, \ldots, X_n$ may be viewed for each $n \in \mathbb{N}$ as the gains in ducats (losses when negative) of n gamblers Paul$_1$, Paul$_2$, …, Paul$_n$, each playing one trial of the same game of chance. As in the preceding chapters, our Pauls may not trust their own luck and, before they play, they may agree to use a *pooling strategy* $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$, where the components are non-negative and add to unity. Using this strategy, Paul$_1$ receives $p_{1,n}X_1 + p_{2,n}X_2 + \cdots + p_{n,n}X_n$ ducats, Paul$_2$ receives $p_{n,n}X_1 + p_{1,n}X_2 +$

$\cdots + p_{n-1,n}X_n$ ducats, ..., and Paul$_n$ receives $p_{2,n}X_1 + p_{3,n}X_2 + \cdots + p_{1,n}X_n$ ducats. Then all the individual winnings are pooled and this rotating system is fair to every Paul since their pooled winnings are equally distributed. The prototypes of such games are the generalized St. Petersburg(α, p) games, since, as we have seen in Chapter 2, in this case X belongs to the domain of geometric partial attraction of a semistable law, defined in (2.2); or this was proved directly by (3.6) in [32].

Returning now to the general situation when $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$, our first main interest in this paper is the asymptotic distribution of the random variable

$$S_{\alpha,\boldsymbol{p}_n} = \sum_{j=1}^{n} \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} X_j - \sum_{j=1}^{n} \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} \int_{p_{j,n}}^{1-p_{j,n}} Q(s)\,\mathrm{d}s, \tag{3.8}$$

where the slowly varying function $l(\cdot)$ is from the representation (3.6) of the quantile function Q corresponding to F. We consider a sequence of strategies $\{\boldsymbol{p}_n\}$ that satisfies the asymptotic negligibility condition $\overline{p}_n = \max\{p_{j,n} \colon j = 1, 2, \ldots, n\} \to 0$.

The main result in this paper is Theorem 3.1 below, a merge theorem for $S_{\alpha,\boldsymbol{p}_n}$ in (3.8). The phenomenon of *merge* takes place when neither of two sequences of distributions converge weakly, but the Lévy or supremum distance between the n-th terms goes to zero as $n \to \infty$ along the entire sequence $\mathbb{N}$.

These linear combinations $S_{\alpha,\boldsymbol{p}_n}$ belong to a real pooling strategy only when $\alpha = 1$ and the slowly varying function $l(\cdot) \equiv 1$ in (3.6). The equivalent Theorem 3.2 contains a satisfactory version after a simple transformation. A surprising consequence is that for some sequences of strategies $\{\boldsymbol{p}_n\}$ ordinary asymptotic distributions of $S_{\alpha,\boldsymbol{p}_n}$ exist as $n \to \infty$ along the entire $\mathbb{N}$. In Section 3 we investigate merge on $\mathbb{R}$ in general and obtain necessary and sufficient Fourier–analytic conditions under weak assumptions. All the proofs are placed in Section 4.

3.2. Merging semistable approximations

Let $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ be semistable with exponent $\alpha \in (0,2)$ as before. For $\psi \in \Psi$ and $\lambda > 0$, let ${}_\lambda\psi(s) = \psi(s/\lambda)$ and put $\psi_j^{\alpha,\lambda}(s) = \lambda^{-1/\alpha}{}_\lambda\psi_j^\alpha(s) = -M_j(s/\lambda)s^{-1/\alpha}$, $s > 0$, where the functions M_j are from (3.3), $j = 1, 2$. Introduce

$$V_{\alpha,\lambda}(M_1, M_2) = V(\psi_1^{\alpha,\lambda}, \psi_2^{\alpha,\lambda}, 0) \text{ and } \mathbf{E}(\mathrm{e}^{\mathrm{i}tV_{\alpha,\lambda}(M_1,M_2)}) = \mathrm{e}^{y_{\alpha,\lambda}(t)},\ t \in \mathbb{R}, \tag{3.9}$$

and notice the identity $V_{\alpha,\lambda}(M_1, M_2) = \lambda^{-1/\alpha} V({}_\lambda\psi_1^\alpha, {}_\lambda\psi_2^\alpha, 0)$. The notation is the same as in [13] with two important exceptions. The random variable that belongs to λ here, belongs to λ^{-1} there ([13], p. 96). The other exception is the function γ_x mentioned before. The reason for the deviation is that for generalized St. Petersburg games our theorems here must reduce to the merge theorems in Chapter 2 and [10].

We have already seen in Chapter 2 that the *circular convergence* plays an important role at the limiting behavior of the sums, so it is natural to extend its definition. For a given $\mathbf{c} > 1$ we say that the sequence $\{u_n\}_{n=1}^\infty \subset \mathbb{R}$ converges circularly to $u \in (\mathbf{c}^{-1}, 1]$, written $u_n \xrightarrow{\text{cir}} u$, if either $u \in (\mathbf{c}^{-1}, 1)$ and $u_n \to u$, or $u = 1$ and the sequence $\{u_n\}$ has limit points $\mathbf{c}^{-1}$ or 1, or both. (For $\mathbf{c} = 1$ the notion $u_n \xrightarrow{\text{cir}} 1$ simply means that $u_n \to 1$.) Let the distribution function $F \in \mathbb{D}_{\text{gp}}(G)$ be such that (3.4) holds along a subsequence $\{k_n\}_{n=1}^\infty$ satisfying (3.5), where $c = \mathbf{c}(G)$; this and nothing else is assumed for Theorems 3.1, 3.2 and the Corollary below. Part of the surprising result in Theorem 1 in [13] is that there are as many different limiting distributions as the continuum along different subsequences:

Theorem. (Csörgő, Megyesi). *If along a subsequence* $\{n_r\}_{r=1}^\infty \subset \mathbb{N}$,

$$\frac{\sum_{j=1}^{n_r} X_j - c_{n_r}}{a_{n_r}} \xrightarrow{\mathcal{D}} W \quad \text{as} \quad r \to \infty \tag{3.10}$$

for a non-degenerate random variable W, *then* $\gamma_{n_r} \xrightarrow{\text{cir}} \kappa \in (\mathbf{c}^{-1}, 1]$ *as* $r \to \infty$, *and the distribution of* W *is necessarily that of an affine linear transformation of* $V_{\alpha,\kappa}(M_1, M_2)$, *namely*

$$W \stackrel{\mathcal{D}}{=} \delta V_{\alpha,\kappa}(M_1, M_2) + d,$$

where

$$\delta = \lim_{r\to\infty} \frac{n_r^{1/\alpha} l(1/n_r)}{a_{n_r}} > 0 \quad \text{and} \quad d = \lim_{r\to\infty} \frac{n_r \int_{n_r^{-1}}^{1-n_r^{-1}} Q(s)\mathrm{d}s}{a_{n_r}}.$$

Conversely, if $\gamma_{n_r} \xrightarrow{\text{cir}} \kappa \in (\mathbf{c}^{-1}, 1]$ *as* $r \to \infty$, *then* (3.10) *holds with* $c_{n_r} = n_r \int_{n_r^{-1}}^{1-n_r^{-1}} Q(s)\,\mathrm{d}s$, $a_{n_r} = n_r^{1/\alpha} l(1/n_r)$ *and* $W = V_{\alpha,\kappa}(M_1, M_2)$.

Now let $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$ be any strategy, so that the components are nonnegative and $\sum_{j=1}^n p_{j,n} = 1$, and for simplicity put $\gamma_{j,n} = \gamma_{1/p_{j,n}}$ if $p_{j,n} > 0$, $j = 1, \ldots, n$. The merging semistable approximation to the distribution functions of $S_{\alpha,\boldsymbol{p}_n}$ in (3.8) is given in the following main result by the distribution functions $G_{\alpha,\boldsymbol{p}_n}(x) = \mathbf{P}\{V_{\alpha,\boldsymbol{p}_n} \le x\}$, $x \in \mathbb{R}$, of random variables $V_{\alpha,\boldsymbol{p}_n}$ that have characteristic functions

$$\text{(3.11)}\quad \mathbf{E}\big(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}\big) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}G_{\alpha,\boldsymbol{p}_n}(x) = \exp\left\{\sum_{j=1}^{n} p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t)\right\}, \quad t \in \mathbb{R},$$

where $y_{\alpha,\gamma_{j,n}}(\cdot)$ is the exponent function in the characteristic function of $V_{\alpha,\gamma_{j,n}}$ in (3.9), explicitly given in the proof of Lemma 3.1 below.

Theorem 3.1. *For any sequence* $\{\boldsymbol{p}_n\}_{n=1}^{\infty}$ *of strategies such that* $\overline{p}_n \to 0$,

$$\sup_{x\in\mathbb{R}} \big|\mathbf{P}\big\{S_{\alpha,\boldsymbol{p}_n} \le x\big\} - G_{\alpha,\boldsymbol{p}_n}(x)\big| \to 0.$$

It follows from the formula (3.11) that for the uniform strategies $\boldsymbol{p}_n^{\diamond} = (1/n, 1/n, \ldots, 1/n)$ the distributional equality $V_{\alpha,\boldsymbol{p}_n} \stackrel{\mathcal{D}}{=} V_{\alpha,\gamma_n}(M_1, M_2)$ holds, and hence Theorem 3.1 reduces to the most important special case of full sums in Theorem 2 in [13].

As noted before, there is real pooling of winnings only if $\alpha = 1$ and $l(\cdot) \equiv 1$ when the sum of the coefficients in (3.8) is 1. However, by a transformation we obtain a version of Theorem 3.1 that is satisfactory in this respect. This transformation is a generally implicit extension of that given in Chapter 2. The function $f(s) = s^{1/\alpha}/l(s)$ in (3.6) is regularly varying of order $1/\alpha$ at zero, and hence by general theory ([4], p. 23) it is asymptotically equivalent to a non-decreasing function. Therefore, to state Theorem 3.2 below, we may and do assume that $f(s) = s^{1/\alpha}/l(s)$ is itself non-decreasing and hence, by monotonicity, its inverse function $g(s)$ exists and it is also non-decreasing for s in a right neighborhood of zero. Then, if $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$ is an arbitrary strategy, consider

$$q_{j,n} = \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})}\left(\sum_{k=1}^{n} \frac{p_{k,n}^{1/\alpha}}{l(p_{k,n})}\right)^{-1} = \frac{f(p_{j,n})}{\sum_{k=1}^{n} f(p_{k,n})}, \quad j = 1, 2, \ldots, n.$$

Then, clearly, $\boldsymbol{q}_n = (q_{1,n}, q_{2,n}, \ldots, q_{n,n})$ is a strategy. We need a one-to-one correspondence, that is, we have to determine $\boldsymbol{p}_n$ in terms of $\boldsymbol{q}_n$. Multiplying the defining equation by $\sum_{k=1}^{n} f(p_{k,n})$ and applying the inverse function $g(\cdot)$, we get the equation $g(q_{j,n} \sum_{k=1}^{n} f(p_{k,n})) = p_{j,n}$, so that summing for j we have $\sum_{j=1}^{n} g(q_{j,n} \sum_{k=1}^{n} f(p_{k,n})) = 1$. The monotonicity of $g(\cdot)$ implies that for a given strategy $\boldsymbol{q}_n$ there exists a *unique* constant $A_{\boldsymbol{q}_n} > 0$ for which $\sum_{j=1}^{n} g(q_{j,n} A_{\boldsymbol{q}_n}) = 1$, so that $A_{\boldsymbol{q}_n} = \sum_{k=1}^{n} f(p_{k,n})$. Thus $p_{j,n} = g(q_{j,n} A_{\boldsymbol{q}_n})$,

$j = 1, 2, \ldots, n$, that is, the correspondence between $\boldsymbol{p}_n$ and $\boldsymbol{q}_n$ is one-to-one indeed. Now we can define the functions and random variables related to the strategy $\boldsymbol{q}_n$. Set $\nu_{k,n} = \gamma_{1/g(q_{k,n}A_{\boldsymbol{q}_n})}$, $k = 1, \ldots, n$, introduce

$$T_{\alpha,\boldsymbol{q}_n} = A_{\boldsymbol{q}_n} \sum_{k=1}^{n} q_{k,n} X_k - A_{\boldsymbol{q}_n} \sum_{k=1}^{n} q_{k,n} \int_{g(q_{k,n}A_{\boldsymbol{q}_n})}^{1-g(q_{k,n}A_{\boldsymbol{q}_n})} Q(s)\,\mathrm{d}s$$

and let $H_{\alpha,\boldsymbol{q}_n}(\cdot)$ be the semistable distribution function with characteristic function

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}H_{\alpha,\boldsymbol{q}_n}(x) = \exp\left\{ \sum_{k=1}^{n} g(q_{k,n}A_{\boldsymbol{q}_n})\, y_{\alpha,\nu_{k,n}}(t) \right\}.$$

Then a reformulated equivalent version of Theorem 3.1 is

Theorem 3.2. *For any sequence $\{\boldsymbol{q}_n\}_{n=1}^{\infty}$ of strategies such that $g(\overline{q}_n A_{\boldsymbol{q}_n}) \to 0$,*

$$\sup_{x\in\mathbb{R}} \left| \mathbf{P}\left\{ T_{\alpha,\boldsymbol{q}_n} \le x \right\} - H_{\alpha,\boldsymbol{q}_n}(x) \right| \to 0.$$

The strange-looking assumption is needed because the relations $\overline{p}_n \to 0$ and $\overline{q}_n \to 0$ are independent in the sense that neither of them implies the other. This can be seen by easily constructed examples, even in the simplest case $l(\cdot) \equiv 1$.

Now we turn back to the setup in (3.8) and (3.11) and show that for special sequences $\{\boldsymbol{p}_n\}$ the merge in Theorem 3.1 reduces to ordinary limit theorems. Since for $\mathbf{c} = 1$ the approximating distribution is one and the same stable distribution already, we assume that $\mathbf{c} > 1$, in which case our conclusion is truly surprising.

Let $\{n_r\}_{r=1}^{\infty} \subset \mathbb{N}$ be an increasing subsequence and consider the sequence of strategies $\boldsymbol{p}_n = (1/n_r, 1/n_r, \ldots, 1/n_r, 0, 0, \ldots, 0)$ with n_r non-zero elements, where $n_r \le n < n_{r+1}$. This is the same situation as in (3.10), so there exists a limiting distribution for $\{\boldsymbol{p}_n\}_{n=1}^{\infty}$ if and only if it exists in (3.10) along $\{n_r\}_{r=1}^{\infty}$. There may be too many zero components in this type of strategies in the sense that in some of them the proportion of zeros is approximately $1 - \mathbf{c}^{-1}$ if $\lim_{r\to\infty} n_{r+1}/n_r = \mathbf{c}$. The following notion excludes such cases: we call a sequence $\{\boldsymbol{p}_n\}_{n=1}^{\infty}$ of strategies *balanced* if

$$\liminf_{n\to\infty} \frac{\min\{p_{j,n} \colon j = 1, 2, \ldots, n\}}{\max\{p_{j,n} \colon j = 1, 2, \ldots, n\}} > 0.$$

Roughly speaking this condition means that each component is important.

Classical theory says that if a limiting distribution exists for the uniform strategies $\boldsymbol{p}_n^\diamond = (1/n, 1/n, \ldots, 1/n)$, it must be stable. As an essence of semistability, the following corollary claims that semistable limiting distributions can be achieved by such balanced strategies that practically consist of only two different components.

Corollary 3.1. *For an arbitrary* $\kappa \in (\mathbf{c}^{-1}, 1]$ *there exists a balanced sequence* $\{\boldsymbol{p}_n\}_{n=1}^\infty$ *of strategies such that* $S_{\alpha,\boldsymbol{p}_n} \xrightarrow{\mathcal{D}} V_{\alpha,\kappa}(M_1, M_2)$*, where the random variable* $V_{\alpha,\kappa}(M_1, M_2)$ *is defined in* (3.9)*. Moreover, for each* $n \in \{2, 3, \ldots\}$ *the strategy* $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$ *can be constructed in such a way that there are at most two different values among its first* $n-1$ *components.*

It will be clear from the proof that the n-th component $p_{n,n}$, which can have a third different value, is just to make $\boldsymbol{p}_n$ a strategy, that is, to make $\sum_{j=1}^n p_{j,n} = 1$. Thus in fact there are only two different important components.

The difficulties of a closer description of the merging semistable random variables $V_{\alpha,\boldsymbol{p}_n}$ in (3.11) arise from the fact that the asymptotic equality $\gamma_{\mathbf{c}x} \sim \gamma_x$, as $x \to \infty$, for the function γ_x figuring in (3.6) does not reduce to true equality. Nevertheless, (3.7) says that for the special sequence $k_n \equiv \lfloor \mathbf{c}^n \rfloor$ we can define the function γ_x through the sequence $\mathbf{c}^n$ instead of $\lfloor \mathbf{c}^n \rfloor$ and obtain explicitly $\gamma_x = x/\mathbf{c}^{\lceil \log_{\mathbf{c}} x \rceil}$ for all $x > 0$. In this case, when $k_n \equiv \lfloor \mathbf{c}^n \rfloor$, let $V_{\alpha,1}, V_{\alpha,2}, \ldots, V_{\alpha,n}$ be independent copies of $V_{\alpha,1}(M_1, M_2)$. Then with $r_{j,n} = \lceil \log_{\mathbf{c}} p_{j,n}^{-1} \rceil$ and $\gamma_{j,n} = \gamma_{p_{j,n}^{-1}} = (p_{j,n}\mathbf{c}^{r_{j,n}})^{-1}$ as before, for any strategy $\boldsymbol{p}_n$ Lemmas 3.1 and 3.6 below imply the distributional equality

$$\sum_{j=1}^n p_{j,n}^{1/\alpha} V_{\alpha,j} - \sum_{j=1}^n \left(d_{-r_{j,n}} + p_{j,n}\, c_{\gamma_{j,n}}\right) \overset{\mathcal{D}}{=} V_{\alpha,\boldsymbol{p}_n}, \tag{3.12}$$

where the constants c_λ, $\lambda > 0$, and d_m, $m \in \mathbb{Z}$, are also from those lemmas.

3.3. Merge theorems in general

The systematic study of merge was initiated in [19] in the general setup of separable metric spaces. The study there did not get down to the characterization of merge in the Lévy distance on $\mathbb{R}$, and the aim of the present small section is exactly that. Of course, the deep and extended literature on Kolmogorov's uniform limit problem, highlighted by Arak's and Zaitsev's well-known results, deals with merge in the uniform distance ever since Prokhorov's first result in 1955. In our list here, [7] and [13] are also examples for merge in the uniform distance.

In this section $X, X_1, X_2, \ldots$, and $Y, Y_1, Y_2, \ldots$ are real random variables with distribution and characteristic functions $F, F_1, F_2, \ldots, G, G_1, G_2, \ldots$ and $\phi, \phi_1, \phi_2, \ldots, \psi, \psi_1, \psi_2, \ldots$, respectively. If $F_n \Rightarrow G$ denotes weak convergence, that is, $F_n(x) \to G(x)$ at each $x \in C(G)$, where we recall that $C(G)$ is the set of continuity points of G, then of course $F_n \Rightarrow G$ is the definition of $X_n \xrightarrow{\mathcal{D}} Y$ used above, which is equivalent to $L(F_n, G) \to 0$, where $L(\cdot\,,\cdot)$ is Lévy's distance, given by $L(F, G) = \inf\{h > 0\colon G(x-h)-h \leq F(x) \leq G(x+h)+h\}$. Extending this, we say that X_n and Y_n, or their distribution functions F_n and G_n, *merge together* if $L(F_n, G_n) \to 0$.

Here we give necessary and sufficient conditions for merge in terms of characteristic functions under the weak assumption that one of the sequences, $\{Y_n\}$ or equivalently $\{G_n\}$, say, is stochastically compact, meaning that for every subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ there is a further subsequence $\{n_{k_j}\}_{j=1}^{\infty} \subset \{n_k\}_{k=1}^{\infty}$ and a random variable Y, such that $Y_{n_{k_j}} \xrightarrow{\mathcal{D}} Y$, or equivalently $G_{n_{k_j}} \Rightarrow G$ as $j \to \infty$.

Theorem 3.3. *If $\{G_n\}_{n=1}^{\infty}$ is stochastically compact, then $L(F_n, G_n) \to 0$ if and only if $\phi_n(t) - \psi_n(t) \to 0$ for every $t \in \mathbb{R}$.*

The next theorem is the basic tool in the proof of Theorem 3.1. It says that if G_n is absolutely continuous for all $n \in \mathbb{N}$ and the corresponding density functions are uniformly bounded, then even uniform convergence holds under the same conditions.

Theorem 3.4. *Assume that $\{G_n\}_{n=1}^{\infty}$ is stochastically compact and there is a constant $K > 0$ such that $\sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}} |G_n'(x)| \leq K$. Then $F_n(x) - G_n(x) \to 0$ at every $x \in \mathbb{R}$ if and only if $\phi_n(t) - \psi_n(t) \to 0$ at every $t \in \mathbb{R}$. Moreover, if this holds, then in fact the convergence is uniform, so that $\sup_{x\in\mathbb{R}} |F_n(x) - G_n(x)| \to 0$.*

3.4. Proofs

Logic dictates to prove first the general theorems from the preceding section.

Proof of Theorem 3.3. Suppose first that $\phi_n(t) - \psi_n(t) \to 0$ for all $t \in \mathbb{R}$. Let $\{n_k\}_{k=1}^{\infty}$ be any subsequence of $\mathbb{N}$. By compactness there is a further subsequence $\{n_{k_j}\}_{j=1}^{\infty} \subset \{n_k\}_{k=1}^{\infty}$ and a distribution function G such that $G_{n_{k_j}} \Rightarrow G$, so that $\psi_{n_{k_j}}(t) \to \psi(t)$, $t \in \mathbb{R}$, as $j \to \infty$ by continuity theorem. By the triangle inequality and the other direction in the continuity theorem, $F_{n_{k_j}} \Rightarrow G$, and so the triangle inequality for the Lévy metric yields

$L(F_{n_{k_j}}, G_{n_{k_j}}) \to 0$ as $j \to \infty$. Since $\{n_k\}$ was arbitrary, it follows that $L(F_n, G_n) \to 0$. The proof of the converse is similar. ∎

Proof of Theorem 3.4. Necessity is trivial, while the proof of sufficiency in the first statement is similar to the one above: using the uniform boundedness of G_n', one can show that the subsequential weak limits G are continuous, and so weak convergence implies convergence in each point.

To prove the stronger second statement, fix any $\varepsilon \in (0,1)$. Stochastic compactness is tightness, so there exists a $T > 0$ such that $G_n(x) > 1 - \varepsilon$ and $G_n(-x) < \varepsilon$ for all $x > T$ and $n \in \mathbb{N}$, and the uniform boundedness of the densities implies the existence of a subdivision $-T = x_0 < x_1 < \cdots < x_N = T$ such that $\sup_{1 \le k \le N, n \in \mathbb{N}} |G_n(x_k) - G_n(x_{k-1})| < \varepsilon$. Since F_n and G_n merge together at each point, there is a threshold $n_0 \in \mathbb{N}$ such that $\max_{k=0,1,\ldots,N} |F_n(x_k) - G_n(x_k)| < \varepsilon$ if $n \ge n_0$. Then by easy calculation $\sup_{x \in \mathbb{R}} |F_n(x) - G_n(x)| < 2\varepsilon$ for all $n \ge n_0$. ∎

Aiming at Theorem 3.1, first we prove six lemmas. The first is a scaling property that expresses the exponent function $y_{\alpha,\lambda}(\cdot)$ of the characteristic function in (3.9) in terms of $y_{\alpha,1}(\cdot)$, which was used for (3.12) and is needed for Lemmas 3.2 and 3.3.

Lemma 3.1. *For every $\lambda > 0$ we have $y_{\alpha,\lambda}(t) = \lambda\, y_{\alpha,1}(t/\lambda^{1/\alpha}) - \mathrm{i}tc_\lambda$, $t \in \mathbb{R}$, where $c_\lambda = \lambda^{1-1/\alpha} \int_1^{1/\lambda} \left[\psi_2^\alpha(s) - \psi_1^\alpha(s)\right] \mathrm{d}s$.*

Proof. As in (3.1), let L_λ and R_λ denote the Lévy functions of the random variable $V({}_\lambda\psi_1^\alpha, {}_\lambda\psi_2^\alpha, 0)$ defined at (3.9). The inverse relation above (3.1) for the two representations shows that $L_\lambda(x) = \inf\{s : {}_\lambda\psi_1^\alpha(s) \ge x\} = \inf\{s : \psi_1^\alpha(s/\lambda) \ge x\} = \lambda L(x)$, $x < 0$, and similarly $R_\lambda(x) = \lambda R(x)$, $x > 0$, where $L(\cdot) = L_1(\cdot)$ and $R(\cdot) = R_1(\cdot)$. Thus, since $V(\psi_1, \psi_2, \sigma) = W(\psi_1, \psi_2, \sigma) + \theta(\psi_1) - \theta(\psi_2)$ in (3.2),

$$
\begin{aligned}
\mathrm{e}^{y_{\alpha,\lambda}(t)} &= \mathbf{E}\Big(\mathrm{e}^{\mathrm{i}tV_{\alpha,\lambda}(M_1,M_2)}\Big) = \mathbf{E}\Big(\mathrm{e}^{\mathrm{i}\frac{t}{\lambda^{1/\alpha}}V({}_\lambda\psi_1^\alpha, {}_\lambda\psi_2^\alpha, 0)}\Big) = \exp\left\{\mathrm{i}t\frac{\theta({}_\lambda\psi_1^\alpha) - \theta({}_\lambda\psi_2^\alpha)}{\lambda^{1/\alpha}}\right\} \\
&\quad \times \exp\left\{\lambda \int_{-\infty}^0 \beta_{\frac{t}{\lambda^{1/\alpha}}}(x)\,\mathrm{d}L(x) + \lambda \int_0^\infty \beta_{\frac{t}{\lambda^{1/\alpha}}}(x)\,\mathrm{d}R(x)\right\},
\end{aligned}
$$

from which, forcing the exponent $y_{\alpha,1}(t\lambda^{-1/\alpha})$ in,

$$
\begin{aligned}
\mathrm{e}^{y_{\alpha,\lambda}(t)} &= \exp\left\{-\,\mathrm{i}t\frac{\theta({}_\lambda\psi_2^\alpha) - \theta({}_\lambda\psi_1^\alpha)}{\lambda^{1/\alpha}} + \mathrm{i}t\lambda\,\frac{\theta(\psi_2^\alpha) - \theta(\psi_1^\alpha)}{\lambda^{1/\alpha}}\right\} \\
&\times \exp\left\{\lambda\left[\mathrm{i}t\,\frac{\theta(\psi_1^\alpha) - \theta(\psi_2^\alpha)}{\lambda^{1/\alpha}} + \int_{-\infty}^0 \beta_{\frac{t}{\lambda^{1/\alpha}}}(x)\,\mathrm{d}L(x) + \int_0^\infty \beta_{\frac{t}{\lambda^{1/\alpha}}}(x)\,\mathrm{d}R(x)\right]\right\}
\end{aligned}
$$

for all $t \in \mathbb{R}$, which is nothing but $\mathrm{e}^{y_{\alpha,\lambda}(t)} = \mathrm{e}^{-\mathrm{i}tc_\lambda}\mathrm{e}^{\lambda y_{\alpha,1}(t\lambda^{-1/\alpha})}$, where $c_\lambda = \lambda^{-1/\alpha}[\theta({}_\lambda\psi_2^\alpha) - \theta({}_\lambda\psi_1^\alpha) - \lambda\{\theta(\psi_2^\alpha) - \theta(\psi_1^\alpha)\}]$. Now, a somewhat long but straightforward calculation shows that $\theta({}_\lambda\psi) = \lambda\,\theta(\psi) + \lambda\int_1^{1/\lambda}\psi(t)\,\mathrm{d}t$. Further simple calculation then yields the stated form of c_λ. ∎

Next, Lemmas 3.2 and 3.3 establish that the sequence $G_{\alpha,\boldsymbol{p}_n}$ in (3.11) has uniformly bounded densities and is stochastically compact, so that it meets the assumptions of Theorem 3.4. Here $\Gamma(u) = \int_0^\infty v^{u-1}\mathrm{e}^{-v}\,\mathrm{d}v$, $u > 0$, is the usual gamma function.

Lemma 3.2. *For any strategy $\boldsymbol{p}_n$ the inequality*

$$\sup_{x\in\mathbb{R}}\left|G'_{\alpha,\boldsymbol{p}_n}(x)\right| \le \frac{\Gamma(1/\alpha)}{\pi\alpha K_\alpha^{1/\alpha}}$$

holds, where the constant $K_\alpha > 0$ depends only on α.

Proof. It follows from a result of Kruglov(3.13) that $\mathfrak{Re}\,y_{\alpha,1}(t) \le -K_\alpha|t|^\alpha$, $t \in \mathbb{R}$. Then by Lemma 3.1, $\mathfrak{Re}\,y_{\alpha,\lambda}(t) = \lambda\mathfrak{Re}\,y_{\alpha,1}(t\lambda^{-1/\alpha}) \le -\lambda K_\alpha|t|^\alpha\lambda^{-1} = -K_\alpha|t|^\alpha$, for all $\lambda > 0$. Thus the distribution function of the variable in (3.9) and hence also $G_{\alpha,\boldsymbol{p}_n}(\cdot)$ in (3.11) is infinitely many times differentiable. In particular,

$$\begin{aligned}\left|G'_{\alpha,\boldsymbol{p}_n}(x)\right| &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tx}\mathbf{E}\big(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}\big)\,\mathrm{d}t\right| \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{\sum_{k=1}^{n}p_{k,n}\mathfrak{Re}\,y_{\alpha,\gamma_{k,n}}(t)\right\}\mathrm{d}t \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{-K_\alpha|t|^\alpha\}\,\mathrm{d}t = \frac{\Gamma(1/\alpha)}{\pi\alpha K_\alpha^{1/\alpha}}\end{aligned}$$

for all $x \in \mathbb{R}$ by the density inversion formula, proving the lemma. ∎

Lemma 3.3. *For any sequence of strategies $\{\boldsymbol{p}_n\}_{n=1}^\infty$, the sequence of random variables $\{V_{\alpha,\boldsymbol{p}_n}\}_{n=1}^\infty$ is stochastically compact.*

Proof. We rewrite the characteristic function in (3.11) in a form that was used in the St. Petersburg case in [17], p. 984. Let denote $I(A)$ the indicator of the event A and put $T_{\boldsymbol{p}_n}(\gamma) = \sum_{j=1}^n p_{j,n}I(\gamma_{j,n} \le \gamma)$, $0 < \gamma \le 1$. Then we have

$$\mathbf{E}\big(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}\big) = \exp\left\{\sum_{j=1}^{n}p_{j,n}y_{\alpha,\gamma_{j,n}}(t)\right\} = \exp\left\{\int_0^1 y_{\alpha,\gamma}(t)\,\mathrm{d}T_{\boldsymbol{p}_n}(\gamma)\right\}.$$

By the multiplicative periodicity $y_{\alpha,\mathbf{c}\gamma}(t) = y_{\alpha,\gamma}(t)$ and by Lemma 3.1, $y_{\alpha,\gamma}(t)$ is a continuous and bounded function of γ in $(0,1]$ for each fixed $t \in \mathbb{R}$, while $T_{\boldsymbol{p}_n}$ is like an empirical distribution function with support contained in $[0,1]$. Since no mass can escape, the lemma follows by an application of the Helly selection theorem. ■

The following measure-theoretic lemma is also important in the proof of Theorem 3.1. It allows to pass on from subsequences to the entire sequence $\mathbb{N}$. Measurability and almost everywhere assumptions are meant in the usual Lebesgue sense and $\mathrm{mes}\{\cdot\}$ stands for Lebesgue measure and $\xrightarrow{\mathrm{mes}}$ denotes convergence in measure.

Lemma 3.4. *Let $q_n : I \to \mathbb{R}$ be sequence of measurable functions, $n \in \mathbb{N}$, and $\delta : \mathbb{N} \to \Lambda$ a sequence taking values in Λ, where $I \subset \mathbb{R}$ and $\Lambda \subset \mathbb{R}$ are compact intervals, and let $\nu_\lambda : I \to \mathbb{R}$ be a set of measurable functions, $\lambda \in \Lambda$. Suppose that if $\lim_{r\to\infty} \delta(n_r) = \lambda$ for a subsequence $\{n_r\}_{r=1}^\infty \subset \mathbb{N}$, then $q_{n_r}(s) - \nu_{\delta(n_r)}(s) \to 0$ for almost every $s \in I$ as $r \to \infty$. Then $q_n(\cdot) - \nu_{\delta(n)}(\cdot) \xrightarrow{\mathrm{mes}} 0$, that is, $\mathrm{mes}\{s \in I : |q_n(s) - \nu_{\delta(n)}(s)| > \varepsilon\} \to 0$ for every $\varepsilon > 0$.*

Proof. Fix any $\varepsilon > 0$ and let $A_n(\varepsilon) = \{s : |q_n(s) - \nu_{\delta(n)}(s)| > \varepsilon\}$. We have to prove that $\mathrm{mes}\{A_n(\varepsilon)\} \to 0$. Let $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ be any subsequence. Since Λ is compact, by the Bolzano–Weierstrass theorem there is a further subsequence $\{n_{k_l}\}_{l=1}^\infty \subset \{n_k\}_{k=1}^\infty$ such that $\delta(n_{k_l}) \to \lambda$ for some $\lambda \in \Lambda$ as $l \to \infty$. By assumption we have $q_{n_{k_l}}(s) - \nu_{\delta(n_{k_l})}(s) \to 0$ as $l \to \infty$ for almost all $s \in I$. Then by Egorov's theorem there exists a measurable set $E \subset I$ on which the convergence is uniform and $\mathrm{mes}\{I \setminus E\} < \varepsilon$. Thus $A_{n_{k_l}}(\varepsilon) \subset I \setminus E$ and so $\mathrm{mes}(A_{n_{k_l}}(\varepsilon)) < \varepsilon$ for all l large enough. Since $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ was arbitrary, the proof is complete. ■

Lemma 3.4 will be used in a slightly different situation. The compact interval Λ will be the 'circle' $(\mathbf{c}^{-1}, 1]$ as the points $\mathbf{c}^{-1}$ and 1 are identified, and the convergence relation $\lim_{r\to\infty} \delta(n_r) = \lambda$ will be replaced by the corresponding $\delta(n_r) \xrightarrow{\mathrm{cir}} \lambda$ as $r \to \infty$. Obviously, the lemma remains true in this setup.

Lemma 3.5. *If $\{n_r\}_{r=1}^\infty \subset \mathbb{N}$ is a subsequence such that $\gamma_{n_r} \xrightarrow{\mathrm{cir}} \kappa \in (\mathbf{c}^{-1}, 1]$ as $r \to \infty$, then*

$$\frac{Q_+(s/n_r)}{n_r^{1/\alpha} l(1/n_r)} - \psi_1^{\alpha,\gamma_{n_r}}(s) \to 0, \quad s \in C(\psi_1^{\alpha,\kappa}),$$

$$-\frac{Q(1 - s/n_r)}{n_r^{1/\alpha} l(1/n_r)} - \psi_2^{\alpha,\gamma_{n_r}}(s) \to 0, \quad s \in C(\psi_2^{\alpha,\kappa})$$

as $r \to \infty$.

Proof. It is shown for the same $\{n_r\}$ in the proof of Theorem 1 in [13] that

$$\frac{Q_+(s/n_r)}{n_r^{1/\alpha} l(1/n_r)} - \psi_1^{\alpha,\kappa}(s) \to 0, \quad s \in C(\psi_1^{\alpha,\kappa}).$$

Since $\psi_1^{\alpha,1} \equiv \psi_1^{\alpha,\mathbf{c}^{-1}}$, the scaling property $\psi_1^{\alpha,\lambda}(s) = \lambda^{-1/\alpha}\psi_1^{\alpha,1}(s/\lambda)$ above (3.9) implies that $\psi_1^{\alpha,\kappa_n}(s) \to \psi_1^{\alpha,\kappa}(s)$, $s \in C(\psi_1^{\alpha,\kappa})$ whenever $\kappa_n \xrightarrow{\text{cir}} \kappa$. The two properties together give the desired result. The proof of the second statement is analogous. ■

The following general lemma is in fact the semistable property, which is used in this paper only for the proof of (3.12). It goes back to Lévy, and the well-known proof is just patient calculation. (In fact, a certain converse is also true.)

Lemma 3.6. *If* $\mathrm{e}^{y_\alpha(\cdot)}$ *is a semistable characteristic function of exponent* $\alpha \in (0,2)$ *and* $c > 0$ *is a multiplicative period of the functions* M_1 *and* M_2 *in* (3.3), *then* $y_\alpha(c^{m/\alpha}t) = c^m y_\alpha(t) + \mathrm{i}td_m$, $t \in \mathbb{R}$, *for every* $m \in \mathbb{Z}$, *where the constants* $d_m \in \mathbb{R}$ *depend on the distribution.*

Proof of Theorem 3.1. By Lemmas 3.2 and 3.3 the sequence $\{V_{\alpha,\boldsymbol{p}_n}\}$ is stochastically compact and their densities are uniformly bounded. Thus by Theorem 3.4 it suffices to prove that $\Delta_{\alpha,\boldsymbol{p}_n}(t) := |\mathbf{E}(\mathrm{e}^{\mathrm{i}tS_{\alpha,\boldsymbol{p}_n}}) - \mathbf{E}(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}})| \to 0$ at each $t \in \mathbb{R}$.

Fixing $t \neq 0$ and setting

$$\mu(\boldsymbol{p}_n) = \sum_{j=1}^{n} \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} \int_{p_{j,n}}^{1-p_{j,n}} Q(s)\,\mathrm{d}s =: \sum_{j=1}^{n} \mu_{j,n}, \tag{3.13}$$

by (3.8) and (3.11) we can write

$$\begin{aligned}
\Delta_{\alpha,\boldsymbol{p}_n}(t) &= \left| \prod_{j=1}^n \mathbf{E}\left(\exp\left\{ \mathrm{i}t \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} X_j \right\} \right) \mathrm{e}^{-\mathrm{i}t\mu(\boldsymbol{p}_n)} - \exp\left\{ \sum_{j=1}^n p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) \right\} \right| \\
&= \left| \prod_{j=1}^n \big(1 + y_{j,n}(t)\big) - \exp\left\{ \sum_{j=1}^n p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) + \mathrm{i}t\mu(\boldsymbol{p}_n) \right\} \right| \\
&\le \left| \prod_{j=1}^n \big(1 + y_{j,n}(t)\big) - \exp\left\{ \sum_{j=1}^n y_{j,n}(t) \right\} \right| \\
&\quad + \left| \exp\left\{ \sum_{j=1}^n y_{j,n}(t) \right\} - \exp\left\{ \sum_{j=1}^n p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) + \mathrm{i}t\mu(\boldsymbol{p}_n) \right\} \right| \\
&\le \left| \exp\left\{ \sum_{j=1}^n \Big[\log\big(1 + y_{j,n}(t)\big) - y_{j,n}(t) \Big] \right\} - 1 \right| \\
&\quad + \left| \exp\left\{ \sum_{j=1}^n \Big[y_{j,n}(t) - p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) - \mathrm{i}t\mu_{j,n} \Big] \right\} - 1 \right|,
\end{aligned}$$

where

$$y_{j,n}(t) = \mathbf{E}\left(\exp\left\{ \mathrm{i}t \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} X_j \right\} - 1 \right) = \int_0^1 \left[\exp\left\{ \mathrm{i}t \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} Q(s) \right\} - 1 \right] \mathrm{d}s. \tag{3.14}$$

Notice that $y_{j,n}(t) \to 0$ for all $j = 1, \ldots, n$ by the condition $\overline{p}_n \to 0$, and so the logarithms are well defined for all n large enough; in fact for our fixed $t \neq 0$ we will use a threshold $n_t \in \mathbb{N}$ such that $|y_{j,n}(t)| \le 1/2$, $j = 1, \ldots, n$, for all $n \ge n_t$. We must prove that

$$\sum_{j=1}^n I_{j,n}(t) := \sum_{j=1}^n \big| \log\big(1 + y_{j,n}(t)\big) - y_{j,n}(t) \big| \to 0 \tag{3.15}$$

and

$$\sum_{j=1}^n \big[y_{j,n}(t) - p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) - \mathrm{i}t\mu_{j,n} \big] \to 0. \tag{3.16}$$

First we consider (3.15). Expanding the logarithm, for all $n \ge n_t$ we obtain

$$\begin{aligned}
I_{j,n}(t) &= \left| \sum_{l=2}^\infty (-1)^{l+1} \frac{y_{j,n}^l(t)}{l} \right| \le \frac{|y_{j,n}(t)|^2}{2} \sum_{l=0}^\infty \big| y_{j,n}(t) \big|^l = \frac{|y_{j,n}(t)|^2}{2\{1 - |y_{j,n}(t)|\}} \\
&\le |y_{j,n}(t)|^2 \le p_{j,n} \left[\frac{1}{\sqrt{p_{j,n}}} \int_0^1 \left| \exp\left\{ \mathrm{i}t \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} Q(s) \right\} - 1 \right| \mathrm{d}s \right]^2
\end{aligned}$$

by (3.14). Since $\sum_{j=1}^n p_{j,n} = 1$, it is enough to show that

$$f_\alpha(x) := \frac{1}{\sqrt{x}} \int_0^1 \left| \mathrm{e}^{\mathrm{i}tQ(s)x^{1/\alpha}/l(x)} - 1 \right| \mathrm{d}s \to 0 \quad \text{as} \quad x \downarrow 0 \tag{3.17}$$

where $x \in (0,1)$ in general. Since $|\mathrm{e}^{\mathrm{i}u} - 1| \le \min\{2, u\}$, $u \in \mathbb{R}$, we see that

$$\int_0^1 \left| \mathrm{e}^{\mathrm{i}tQ(s)x^{1/\alpha}/l(x)} - 1 \right| \mathrm{d}s \le \int_0^x 2\,\mathrm{d}s + t\,\frac{x^{1/\alpha}}{l(x)} \int_x^{1-x} |Q(s)|\,\mathrm{d}s + \int_{1-x}^1 2\,\mathrm{d}s.$$

Megyesi(3.14), p. 423, proved that for h_0 small enough there exist constants $c_j > 0$ such that $\sup_{s\in(0,h_0]} |M_j(\gamma_{s^{-1}}^{-1}) + h_j(s)| \le c_j$, where $M_j(\cdot)$ and $h_j(\cdot)$ are from (3.6), and we choose c_j so large that the inequalities $\sup_{s\in(0,\infty)} M_j(s) \le c_j$ also hold, $j = 1,2$. Further restrictions on h_0 will be introduced as we go along. Then by (3.6),

$$\big|Q_+(s)\big| \le c_1\,\frac{l(s)}{s^{1/\alpha}} \quad \text{and} \quad \big|Q(1-s)\big| \le c_2\,\frac{l(s)}{s^{1/\alpha}}, \quad 0 < s \le h_0,$$
$$\text{and} \quad \psi_j^{\alpha,\lambda}(s) \le \frac{c_j}{s^{1/\alpha}}, \quad s > 0, \quad j = 1,2, \quad \text{for all} \quad \lambda > 0. \tag{3.18}$$

Hence $\int_x^{h_0} \big|Q_+(s)\big|\,\mathrm{d}s \le c_1 \int_x^{h_0} l(s)s^{-1/\alpha}\,\mathrm{d}s$. Here we take $h_0 > 0$ be so small that $l(\cdot)$ is locally bounded on $(0,h_0)$, that is, $l(\cdot)$ is bounded on (ε, h_0) for each $\varepsilon > 0$. Note that $l(1/v)$, as a function of v, is slowly varying at infinity. We now apply Karamata's theorem ([4], pp. 26–27) and accordingly separate three cases of α.

If $\alpha < 1$ then $\frac{1}{\alpha} - 2 > -1$, and so we have the asymptotic inequality

$$\int_x^{h_0} \frac{l(s)}{s^{1/\alpha}}\,\mathrm{d}s = \int_{1/h_0}^{1/x} v^{\frac{1}{\alpha}-2}\, l(1/v)\,\mathrm{d}v \sim \frac{\alpha}{1-\alpha}\, x^{1-\frac{1}{\alpha}}\, l(x) \quad \text{as} \quad x \downarrow 0,$$

where we write $f(u) \sim g(u)$ if $\lim_{u\to\infty} f(u)/g(u) = 1$, and hence, as $x \downarrow 0$,

$$\begin{aligned} f_\alpha(x) &\le 4\sqrt{x} + t(c_1+c_2)\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} \int_x^{h_0} \frac{l(s)}{s^{1/\alpha}}\,\mathrm{d}s + t\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \\ &= 4\sqrt{x} + t\frac{(c_1+c_2)\alpha}{1-\alpha}\,\sqrt{x}\,\big(1+o(1)\big) + t\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \to 0. \end{aligned}$$

If $\alpha = 1$ then $\frac{1}{\alpha} - 2 = -1$, in which case $l^*(x) = \int_{1/h_0}^{1/x} v^{-1} l(1/v)\,\mathrm{d}v$ is slowly varying at 0, so that, as $x \downarrow 0$,

$$\begin{aligned} f_1(x) &\le 4\sqrt{x} + t\frac{\sqrt{x}}{l(x)}\,(c_1+c_2) \int_x^{h_0} \frac{l(s)}{s}\,\mathrm{d}s + t\frac{\sqrt{x}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \\ &= 4\sqrt{x} + t(c_1+c_2)\sqrt{x}\,\frac{l^*(x)}{l(x)} + t\frac{\sqrt{x}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \to 0. \end{aligned}$$

Finally if $\alpha > 1$ then $2 - \frac{1}{\alpha} > 1$, so that $c_3 := \int_{1/h_0}^{\infty} v^{\frac{1}{\alpha}-2}\, l(1/v)\,\mathrm{d}v < \infty$ and

$$\begin{aligned} f_\alpha(x) &\le 4\sqrt{x} + t(c_1+c_2)\,\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)}\int_x^{h_0}\frac{l(s)}{s^{1/\alpha}}\,\mathrm{d}s + t\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)}\int_{h_0}^{1-h_0}|Q(s)|\,\mathrm{d}s \\ &= 4\sqrt{x} + t(c_1+c_2)c_3\,\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} + t\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)}\int_{h_0}^{1-h_0}|Q(s)|\,\mathrm{d}s \to 0, \end{aligned}$$

as $x \downarrow 0$. Thus (3.17) and, therefore, (3.15) is completely proved.

Now we turn to (3.16). For each $j = 1, 2, \ldots, n$ using the change of variables $s = up_{j,n}$ in (3.13) and in (3.14), we see that

$$\mu_{j,n} = p_{j,n}\int_1^{\frac{1}{p_{j,n}}-1} Q(up_{j,n})\frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})}\,\mathrm{d}u \tag{3.19}$$

and

$$\begin{aligned} y_{j,n}(t) &= p_{j,n}\int_0^{1/p_{j,n}}\Big(\exp\Big\{\mathrm{i}tQ(up_{j,n})p_{j,n}^{1/\alpha}/l(p_{j,n})\Big\}-1\Big)\,\mathrm{d}u \\ &= p_{j,n}\Bigg\{\int_0^{h_0/p_{j,n}}\Big(\exp\Big\{\mathrm{i}tQ(sp_{j,n})p_{j,n}^{1/\alpha}/l(p_{j,n})\Big\}-1\Big)\,\mathrm{d}s \\ &\quad + \int_{h_0/p_{j,n}}^{(1-h_0)/p_{j,n}}\Big(\exp\Big\{\mathrm{i}tQ(sp_{j,n})p_{j,n}^{1/\alpha}/l(p_{j,n})\Big\}-1\Big)\,\mathrm{d}s \\ &\quad + \int_0^{h_0/p_{j,n}}\Big(\exp\Big\{\mathrm{i}tQ(1-sp_{j,n})p_{j,n}^{1/\alpha}/l(p_{j,n})\Big\}-1\Big)\,\mathrm{d}s\Bigg\}. \end{aligned} \tag{3.20}$$

Therefore, (3.16) to be proved is equivalent to $\sum_{j=1}^{n} p_{j,n}J_{j,n}(t) \to 0$, where

$$\begin{aligned} J_{j,n}(t) = &\int_0^{\frac{1}{p_{j,n}}}\left[\exp\left\{\mathrm{i}t\frac{Q(sp_{j,n})p_{j,n}^{1/\alpha}}{l(p_{j,n})}\right\}-1\right]\mathrm{d}s - y_{\alpha,\gamma_{j,n}}(t) \\ &-\mathrm{i}t\int_1^{\frac{1}{p_{j,n}}-1}\frac{Q(sp_{j,n})p_{j,n}^{1/\alpha}}{l(p_{j,n})}\,\mathrm{d}s\,. \end{aligned}$$

Since $\sum_{j=1}^{n} p_{j,n} = 1$ and $\overline{p}_n \to 0$, it suffices to show that

$$h_\alpha(x) \to 0 \quad \text{as} \quad x \downarrow 0, \tag{3.21}$$

where

$$h_\alpha(x) = \int_0^{\frac{1}{x}}\left[\exp\left\{\mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)}\right\}-1\right]\mathrm{d}s - y_{\alpha,\gamma_{1/x}}(t) - \mathrm{i}t\int_1^{\frac{1}{x}-1}Q(sx)\frac{x^{1/\alpha}}{l(x)}\,\mathrm{d}s.$$

Now we rewrite the characteristic function of $G_{\alpha,\boldsymbol{p}_n}(\cdot)$ in the theorem. By (3.1),

$$\begin{aligned}\int_0^\infty \beta_t(\psi_1^{(\alpha,\lambda)}(s))\,\mathrm{d}s &= \int_0^1 \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1\right]\mathrm{d}s - \mathrm{i}t\int_0^1 \frac{\psi_1^{\alpha,\lambda}(s)}{1+\left\{\psi_1^{\alpha,\lambda}(s)\right\}^2}\,\mathrm{d}s \\ &\quad + \int_1^\infty \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\lambda}(s)\right]\mathrm{d}s \\ &\quad + \mathrm{i}t\int_1^\infty \left[\psi_1^{\alpha,\lambda}(s) - \frac{\psi_1^{\alpha,\lambda}(s)}{1+\left\{\psi_1^{\alpha,\lambda}(s)\right\}^2}\right]\mathrm{d}s \\ &= \int_0^1 \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1\right]\mathrm{d}s + \int_1^\infty \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\lambda}(s)\right]\mathrm{d}s \\ &\quad - \mathrm{i}t\theta\left(\psi_1^{\alpha,\lambda}\right),\end{aligned}$$

where $\theta(\psi)$ as above (3.2). With the analogous form of other integral we finally get

$$\begin{aligned}y_{\alpha,\lambda}(t) &= \int_0^1 \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1\right]\mathrm{d}s + \int_1^\infty \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\lambda}(s)\right]\mathrm{d}s \\ &\quad + \int_0^1 \left[\mathrm{e}^{\mathrm{i}t\{-\psi_2^{\alpha,\lambda}(s)\}} - 1\right]\mathrm{d}s + \int_1^\infty \left[\mathrm{e}^{\mathrm{i}t\{-\psi_2^{\alpha,\lambda}(s)\}} - 1 - \mathrm{i}t\{-\psi_2^{\alpha,\lambda}(s)\}\right]\mathrm{d}s.\end{aligned}$$

Using this, (3.19) and (3.20), we obtain

$$
\begin{aligned}
h_\alpha(x) \;=\; & \int_0^1 \Big[\Big(\exp\Big\{ \mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)} \Big\} - 1 \Big) - \Big(\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)} - 1 \Big) \Big] \mathrm{d}s \\
& + \int_1^{h_0/x} \Big[\exp\Big\{ \mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)} \Big\} - 1 - \mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)} \\
& \qquad - \Big(\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s) \Big) \Big] \mathrm{d}s \\
& + \int_{h_0/x}^{(1-h_0)/x} \Big[\exp\Big\{ \mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)} \Big\} - 1 - \mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)} \Big] \mathrm{d}s \\
& + \int_0^1 \Big[\Big(\exp\Big\{ \mathrm{i}tQ(1-sx)\frac{x^{1/\alpha}}{l(x)} \Big\} - 1 \Big) - \Big(\mathrm{e}^{-\mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s)} - 1 \Big) \Big] \mathrm{d}s \\
& + \int_1^{h_0/x} \Big[\exp\Big\{ \mathrm{i}tQ(1-sx)\frac{x^{1/\alpha}}{l(x)} \Big\} - 1 - \mathrm{i}tQ(1-sx)\frac{x^{1/\alpha}}{l(x)} \\
& \qquad - \Big(\mathrm{e}^{-\mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s)} - 1 + \mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s) \Big) \Big] \mathrm{d}s \\
& - \int_{h_0/x}^{\infty} \Big[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s) + \mathrm{e}^{-\mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s)} - 1 + \mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s) \Big] \mathrm{d}s \\
=:\; & h_{\alpha,1}(x) + h_{\alpha,2}(x) + h_{\alpha,3}(x) + h_{\alpha,4}(x) + h_{\alpha,5}(x) - h_{\alpha,6}(x).
\end{aligned}
$$

Using the inequality $|\mathrm{e}^{\mathrm{i}u} - 1 - \mathrm{i}u| \leq u^2/2$, $u \in \mathbb{R}$, and then the bounds $\{\psi_j^{\alpha,\gamma_{1/x}}(s)\}^2 \leq c_j^2/s^{2/\alpha}$, $j = 1, 2$, established in (3.18), we see that $|h_{\alpha,6}(x)| \leq 2^{-1}(c_1^2 + c_2^2)\, t^2 \int_{h_0/x}^{\infty} s^{-2/\alpha}\,\mathrm{d}s \to 0$ as $x \downarrow 0$. Also, with the substitution $sx = y$,

$$
|h_{\alpha,3}(x)| \leq \int_{h_0/x}^{(1-h_0)/x} \frac{t^2 Q^2(sx) x^{2/\alpha}}{l^2(x)}\,\mathrm{d}s = \frac{x^{\frac{2}{\alpha}-1}}{l^2(x)}\, t^2 \int_{h_0}^{1-h_0} Q^2(y)\,\mathrm{d}y \to 0 \;\text{ as }\; \downarrow 0.
$$

Clearly, $h_{\alpha,1}(\cdot)$ and $h_{\alpha,4}(\cdot)$ behave analogously and can be handled the same way, and $h_{\alpha,2}(\cdot)$ and $h_{\alpha,5}(\cdot)$ can also be handled the same way. Hence we deal only with $h_{\alpha,1}(\cdot)$ and $h_{\alpha,2}(\cdot)$. First note that Lemmas 3.4 and 3.5 together imply

$$
\mathrm{mes}\Big\{ 0 \leq s \leq N \colon \Big| \frac{Q_+(s/n)}{n^{1/\alpha} l(1/n)} - \psi_1^{\alpha,\gamma_n}(s) \Big| > \varepsilon \Big\} \to 0 \quad \text{for all} \quad \varepsilon > 0,
$$

convergence in measure on $[0, N]$ for each $N > 0$. Using the monotonicity of $\psi_1^{\alpha,\gamma_{1/x}}(\cdot)$ and $Q(\cdot)$, we show that in this convergence $n^{-1} \downarrow 0$ can be extended to $x \downarrow 0$. To this end, consider any $x_n \downarrow 0$ such that $\gamma_{1/x_n} \xrightarrow{\text{cir}} \kappa \in (\mathbf{c}^{-1}, 1]$. Then also $\gamma_{\lfloor 1/x_n \rfloor} \xrightarrow{\text{cir}} \kappa$ and $\gamma_{\lceil 1/x_n \rceil} \xrightarrow{\text{cir}} \kappa$, so that, according to the proof of Lemma 3.5, $Q_+(s/y_n)/\{y_n^{1/\alpha} l(1/y_n)\} \to \psi_1^{\alpha,\kappa}(s)$ and $\psi_1^{\alpha,\gamma_{y_n}}(s) \to \psi_1^{\alpha,\kappa}(s)$, $s \in C(\psi_1^{\alpha,\kappa})$, where y_n can be chosen in both convergence relations as $1/x_n$, $\lceil 1/x_n \rceil$ and $\lfloor 1/x_n \rfloor$. Using that $Q_+(s/\lceil 1/x_n \rceil) \le Q_+(sx_n) \le Q_+(s/\lfloor 1/x_n \rfloor)$, $l(1/\lfloor 1/x_n \rfloor)/l(x_n) \to 1$ and $l(1/\lceil 1/x_n \rceil)/l(x_n) \to 1$, we get by standard manipulation that $\{Q_+(sx_n)x_n^{1/\alpha}/l(x_n)\} - \psi_1^{\alpha,\gamma_{1/x_n}}(s) \to 0$ for all $s \in C(\psi_1^{\alpha,\kappa})$. This implies by Lemma 3.4 that

$$\mathrm{mes}\left\{0 \le s \le N \colon \left| \frac{Q_+(sx)\, x^{1/\alpha}}{l(x)} - \psi_1^{\alpha,\gamma_{1/x}}(s) \right| > \varepsilon \right\} \to 0 \quad \text{for all} \quad \varepsilon > 0,$$

as $x \downarrow 0$. We note that if the functions ψ_j^α, $j = 1, 2$, in (3.3) are continuous, then Lemma 3.4 is needless because convergence holds pointwise.

Thus, towards the proof of (3.21), we showed that in the integrands in $h_{\alpha,1}(\cdot)$ and $h_{\alpha,2}(\cdot)$ go to 0 in measure as $x \downarrow 0$ on each interval $[0, N]$. Thus, it suffices to find common integrable bounds. For the first integral the function 2 does the job, so that $h_{\alpha,1}(x) \to 0$ and $h_{\alpha,4}(x) \to 0$ as $x \downarrow 0$. For the second, by (3.18) we have

$$\begin{aligned} &\left| \exp\left\{ \frac{\mathrm{i}tQ(sx)x^{1/\alpha}}{l(x)} \right\} - 1 - \frac{\mathrm{i}tQ(s\,x)x^{1/\alpha}}{l(x)} \right| + \left| \mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s) \right| \\ &\qquad \le t^2 \frac{Q^2(sx)x^{2/\alpha}}{l^2(x)} + t^2 \{\psi_1^{\alpha,\gamma_{1/x}}(s)\}^2 \le t^2 \frac{Q^2(sx)x^{2/\alpha}}{l^2(x)} + t^2 \frac{c_1^2}{s^{2/\alpha}}, \end{aligned}$$

and the second term is integrable on $[1, \infty)$. For the first term we need Potter's theorem ([4], p. 25), which for the function $l_\infty(y) = l(1/y)$, $y \ge 1$, slowly varying at infinity, states that for each $\delta > 0$ and $A > 1$ there is a $K = K(A, \delta)$ such that

$$\frac{l_\infty(y)}{l_\infty(z)} \le A \max\left\{ \left(\frac{y}{z}\right)^\delta, \left(\frac{z}{y}\right)^\delta \right\}, \quad y, z > K.$$

Take $A = 2$ and $\delta = (2\alpha)^{-1} - 4^{-1}$ and let $h_0 < 1/K(2, \delta)$. Then for $x < h_0$ and $s \in [1, h_0/x]$ we have $\{l(sx)/l(x)\} \le 2 \max\{s^\delta, s^{-\delta}\} = 2s^\delta$, and so, first by (3.18),

$$\left| \frac{Q_+^2(sx)x^{2/\alpha}}{l^2(x)} \right| \le c_1^2 \frac{l^2(sx)}{(sx)^{2/\alpha}} \frac{x^{2/\alpha}}{l^2(x)} = c_1^2 s^{-\frac{1}{2} - \frac{1}{\alpha}} \left(\frac{l(sx)}{l(x)s^\delta} \right)^2 \le 4c_1^2 s^{-\frac{1}{2} - \frac{1}{\alpha}},$$

which is integrable on $[1, \infty)$. Therefore, $h_{\alpha,2}(x) \to 0$ and $h_{\alpha,5}(x) \to 0$ as $x \downarrow 0$, proving (3.16) and hence the theorem. ∎

Proof of Corollary 3.1. We construct a strategy $\boldsymbol{p}_n$ such that $\gamma_{j,n} = \kappa$ for all $j = 1, 2, \ldots, n-1$, and $p_{n,n} \to 0$. Then for the characteristic function

$$\mathbf{E}(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}) = \exp\left\{ \sum_{j=1}^n p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) \right\} = \mathrm{e}^{y_{\alpha,\kappa}(t)}\, \mathrm{e}^{p_{n,n}\,[y_{\alpha,\gamma_{n,n}}(t) - y_{\alpha,\kappa}(t)]},$$

so that $\mathbf{E}(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}) \to \mathrm{e}^{y_{\alpha,\kappa}(t)}$, $t \in \mathbb{R}$. Since $S_{\alpha,\boldsymbol{p}_n}$ and $V_{\alpha,\boldsymbol{p}_n}$ merge together by Theorem 3.1, we get $S_{\alpha,\boldsymbol{p}_n} \xrightarrow{\mathcal{D}} V_{\alpha,\kappa}(M_1, M_2)$. So it is enough to find such a strategy.

Fix $n \in \mathbb{N}$ sufficiently large to have $k_{n^*-1} < n \leq k_{n^*}$ for $n^* = n^*(n)$, as before (3.6), and put $x_0 = \kappa k_{n^*}$, $x_{-1} = \kappa k_{n^*-1}$ and $x_{+1} = \kappa k_{n^*+1}$. Clearly, $\gamma_{x_j} = \kappa$, $j = 0, \pm 1$. If $x_0 = n$, then the uniform strategy $\boldsymbol{p}_n = (1/n, 1/n, \ldots, 1/n)$ is suitable.

If $x_0 \neq n$, we begin by equating each component to $1/x_0$. Suppose that $x_0 > n$. Then, starting with the first component, we proceed step by step and substitute $1/x_0$ by $1/x_{-1}$, so that the sum of the components is increased at each step. We do this until the sum is still less than 1. Since $n/x_{-1} > 1$, we will not change all components. Finally, increase the last $1/x_0$ to some $p_{n,n} \in (1/x_0, 1/x_{-1})$ that makes the sum 1, and the construction is complete.

For $x_0 < n$ the proof is similar, only we decrease $1/x_0$ by $1/x_{+1}$ at each step. ■

Chapter 4.

Asymptotic normality

4.1. Introduction and results

Let $X, X_1, X_2, \ldots$ be iid random variables with the common distribution function $F(x) = \mathbf{P}\{X \leq x\}$. For each $n \in \mathbb{N} = \{1, 2, \ldots\}$ consider the random variable

$$S_{\boldsymbol{a}_n} = a_{1,n}X_1 + a_{2,n}X_2 + \cdots + a_{n,n}X_n \, ,$$

where $\boldsymbol{a}_n = (a_{1,n}, \ldots, a_{n,n})$ is an arbitrary sequence of weights. We investigate the asymptotic behavior of the weighted sum $S_{\boldsymbol{a}_n}$, therefore it is reasonable to assume that each component is asymptotically negligible, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_{1\leq k\leq n} \mathbf{P}\{|a_{k,n}X_k| \geq \varepsilon\} = \lim_{n\to\infty} \mathbf{P}\{|X| \geq \varepsilon/\overline{a}_n\} = 0,$$

where $\overline{a}_n = \max\{|a_{k,n}| : k = 1, 2, \ldots, n\}$, which holds, if and only if $\overline{a}_n \to 0$, as $n \to \infty$. Therefore from now on we assume that $\overline{a}_n \to 0$.

Since the possible limiting distributions of $S_{\boldsymbol{a}_n}$ are necessarily infinitely divisible, we need the well-known representation of their characteristic functions. As in the previous chapter let Y be an infinitely divisible real random variable with characteristic function $\phi(t) = \mathbf{E}(\mathrm{e}^{\mathrm{i}tY})$ in its Lévy form ([23] p. 70), given for each $t \in \mathbb{R}$ by

$$\phi(t) = \exp\left\{\mathrm{i}t\theta - \frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0} \beta_t(x)\,\mathrm{d}L(x) + \int_{0}^{\infty} \beta_t(x)\,\mathrm{d}R(x)\right\},$$

where

$$\beta_t(x) = \mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}$$

and the constants $\theta \in \mathbb{R}$ and $\sigma \geq 0$ and the functions $L(\cdot)$ and $R(\cdot)$ are uniquely determined: $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $\lim_{x\to-\infty} L(x) = L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and non-decreasing on $(0,\infty)$ with $\lim_{x\to\infty} R(x) = R(\infty) = 0$, such that $\int_{-\varepsilon}^{0} x^2 \mathrm{d}L(x) + \int_0^{\varepsilon} x^2 \mathrm{d}R(x) < \infty$ for every $\varepsilon > 0$.

Our starting point is Theorem 25.1 in [23], which states that for an infinite array of asymptotically negligible, row-wise independent random variables $\{Y_{1,n}, Y_{2,n}, \ldots, Y_{n,n}\}_{n=1}^{\infty}$, with distribution functions $F_{k,n}(x) = \mathbf{P}\{Y_{k,n} \leq x\}$, $x \in \mathbb{R}$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots, n$, the random variable $\sum_{k=1}^{n} Y_{k,n} - c_n$, for an appropriate numerical sequence c_n, converges in distribution to a non-degenerate random variable W, with Lévy functions L and R, and normal component σ, if and only if

$$\begin{aligned} &\textstyle\sum_{k=1}^{n} F_{k,n}(x) \to L(x),\ x < 0,\ x \in C_L\,, \\ &\textstyle\sum_{k=1}^{n} F_{k,n}(x) - 1 \to R(x),\ x > 0,\ x \in C_R\,, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \sum_{k=1}^{n} \left\{ \int_{|x|\leq\varepsilon} x^2 \,\mathrm{d}F_{k,n}(x) - \Big(\int_{|x|\leq\varepsilon} x \,\mathrm{d}F_{k,n}(x) \Big)^2 \right\} \\ &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sum_{k=1}^{n} \left\{ \int_{|x|\leq\varepsilon} x^2 \,\mathrm{d}F_{k,n}(x) - \Big(\int_{|x|\leq\varepsilon} x \,\mathrm{d}F_{k,n}(x) \Big)^2 \right\} = \sigma^2, \end{aligned} \tag{4.2}$$

where for a real function f, C_f denotes its continuity points.

The Lévy functions of the normal distribution are identically 0. Adding the two equations in (4.1) and using Theorem 26.2 in [23] we obtain, that $S_{\boldsymbol{a}_n} - c_n \xrightarrow{\mathcal{D}} Z \sim N(0,1)$ for some appropriate c_n, if and only if for every $\varepsilon > 0$

$$\begin{aligned} &\textstyle\sum_{k=1}^{n} \displaystyle\int_{|a_{k,n} X_k| > \varepsilon} \mathrm{d}\mathbf{P} \to 0 \\ &\textstyle\sum_{k=1}^{n} \displaystyle\left\{ \int_{|a_{k,n} X_k| < \varepsilon} a_{k,n}^2 X_k^2 \,\mathrm{d}\mathbf{P} - \Big(\int_{|a_{k,n} X_k| < \varepsilon} a_{k,n} X_k \,\mathrm{d}\mathbf{P} \Big)^2 \right\} \to 1\,. \end{aligned}$$

It follows immediately from this form that $\sum_{k=1}^{n} a_{k,n} X_k - c_n \xrightarrow{\mathcal{D}} N(0,1)$ if and only if $\sum_{k=1}^{n} |a_{k,n}| X_k - c_n \xrightarrow{\mathcal{D}} N(0,1)$.

In the simplest case, when X has finite variance we obtain the following characterization of convergence:

Theorem 4.1. *Let $X, X_1, X_2, \dots$ be iid random variables with finite variance, and put $\mu = \mathbf{E}(X)$. Then $\overline{a}_n \to 0$ and*

$$\sum_{k=1}^{n} a_{k,n}(X_k - \mu) \xrightarrow{\mathcal{D}} N(0,1)\,,$$

if and only if $\sum_{k=1}^{n} a_{k,n}^2 \to 1/\mathrm{Var}(X)$.

Asymptotic normality of linear combinations is closely related to the following problem: Let $(R_{\nu,1}, R_{\nu,2}, \dots, R_{\nu,N_\nu})$ be a random vector, which takes on the $N_\nu!$ permutations of $(1, \dots, N_\nu)$ with equal probabilities. Consider $\{b_{\nu,i} : 1 \le i \le N_\nu,\ \nu \le 1\}$ and $\{a_{\nu,i} : 1 \le i \le N_\nu,\ \nu \le 1\}$ two double sequence of real numbers. Hájek [25] gave necessary and sufficient condition for the asymptotic normality of the random sum $\sum_{i=1}^{N_\nu} b_{\nu,i}\, a_{\nu,R_{\nu,i}}$. During the proof of his main theorem, as a corollary he obtains Theorem 4.1 here.

Using the language of the previous chapter the reformulation of the theorem is

Corollary 4.1. *Let $X_1, X_2, \dots$ be iid random variables with 0 mean and finite variance. Then for a sequence of strategies $\{\boldsymbol{p}_n\}$, there exists a normalizing sequence a_n, such that*

$$\frac{1}{a_n}\sum_{k=1}^{n} p_{k,n} X_k \xrightarrow{\mathcal{D}} N(0,1)$$

and $\overline{p}_n / a_n \to 0$, if and only if

$$\frac{\overline{p}_n}{\sqrt{\sum_{k=1}^{n} p_{k,n}^2}} \to 0\,,$$

and in this case $a_n = \sqrt{\mathrm{Var}(X) \sum_{k=1}^{n} p_{k,n}^2}$.

An other special case of the weight sequences is the following. Let $X_1, X_2, \dots$ be iid random variables with $\mathbf{E}(X) = 0$ and $\mathbf{E}(X^2) = 1$. Let $\{w_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $w_k \neq 0$ for all k, and put $W_n = w_1^2 + \dots + w_n^2$. The weight sequence is $\boldsymbol{a}_n = (w_1/\sqrt{W_n}, \dots, w_n/\sqrt{W_n})$. Easy computation shows that in this particular case asymptotic negligibility $\overline{a}_n \to 0$ holds if and only if $W_n \to \infty$ and $w_n^2/W_n \to 0$. With no more moment assumptions on X, Fisher [21] proved that $S_{\boldsymbol{a}_n} \xrightarrow{\mathcal{D}} N(0,1)$, if $W_n \to \infty$ and $\limsup_{t\to\infty} \#\{n : W_n/w_n^2 < t\}/t < \infty$, where $\#A$ stands for the cardinality of a set A. It is easy to show that these conditions imply asymptotic negligibility, but the converse is not true. Later Weber [37] found some complicate sufficient conditions for $S_{\boldsymbol{a}_n} \xrightarrow{\mathcal{D}} N(0,1)$, with higher moment assumptions, and these assumptions also imply asymptotic negligibility. As a corollary of Theorem 4.1 we obtain that in this special case asymptotic negligibility immediately implies distributional convergence:

Corollary 4.2. *Let $X_1, X_2, \ldots$, $\{w_n\}_{n=1}^{\infty}$, $\{W_n\}_{n=1}^{\infty}$ and $\boldsymbol{a}_n$ be as above. If $W_n \to \infty$ and $w_n^2/W_n \to 0$, then $S_{\boldsymbol{a}_n} \xrightarrow{\mathcal{D}} N(0,1)$.*

Now assume that the variance is infinite. In this case assumption (4.2), especially in the normal case, becomes simpler, because by [23] p.173

$$\left[\int_{-x}^{x} y \,\mathrm{d}F(y)\right]^2 = o(1) \int_{-x}^{x} y^2 \,\mathrm{d}F(y),$$

where $o(1) \to 0$ as $x \to \infty$.

Recall that the distribution F is in the *domain of attraction* of the α-stable law W, $\alpha \in (0,2]$, written $F \in \mathbb{D}(\alpha)$, if for some centering and norming sequences c_n and a_n (3.4) holds along the whole sequence of natural numbers, that is

$$\frac{1}{a_n}\left[\sum_{k=1}^{n} X_k - c_n\right] \xrightarrow{\mathcal{D}} W\,,$$

where, of course $X_1, X_2, \ldots$, are iid random variables with distribution function F. Moreover, F is in the domain of *partial* attraction of the infinitely divisible random variable W, written $F \in \mathbb{D}_p(W)$, if there exist a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$, and centering and norming sequences c_{k_n} and a_{k_n}, such that (3.4) holds, that is the distributional convergence above, along the subsequence $\{k_n\}_{n=1}^{\infty}$. For an α-stable W we write $\mathbb{D}_p(\alpha)$ instead of $\mathbb{D}_p(W)$.

Theorem 4.2. *Assume that $F \in \mathbb{D}(2)$. If for some weight sequence $\boldsymbol{a}_n$ and centering sequence c_n, $S_{\boldsymbol{a}_n} - c_n \xrightarrow{\mathcal{D}} W$, where W is a nondegenerate random variable, then W is necessarily normal.*

We investigate a particular converse of the theorem above. What can we say about the random variable X, if for some sequence $\boldsymbol{a}_n$ the limit distribution exists, and it is normal?

Theorem 4.3. *Let $X_1, X_2, \ldots$ be iid random variables with common distribution function F. If there exists a weight sequence $\boldsymbol{a}_n$ and a centering numerical sequence c_n, such that $S_{\boldsymbol{a}_n} - c_n \xrightarrow{\mathcal{D}} N(0,1)$, then $F \in \mathbb{D}_p(2)$.*

In a certain sense, according to the latter theorem the distributional convergence through linear combinations is not more general, than along subsequences. The converse is trivially true, as we noted before Corollary 3.1. Indeed, assume that for a given subsequence $\{k_n\}$ (3.4) holds. Then we can define the weight sequence $\boldsymbol{a}_n = (1/a_{k_j}, \dots, 1/a_{k_j}, 0, \dots, 0)$, if $k_j \le n < k_{j+1}$, where the number of $1/ak_j$-s is k_j. Now, obviously $S_{\boldsymbol{a}_n} - c_{k_j}/a_{k_j} \xrightarrow{\mathcal{D}} W$. To exclude such cases we introduce the notion of balancedness for weight sequences: $\{\boldsymbol{a}_n\}_{n=1}^{\infty}$ is *balanced* if

$$\liminf_{n\to\infty} \frac{\min\{|a_{k,n}| \,:\, k = 1, \dots, n\}}{\max\{|a_{k,n}| \,:\, k = 1, \dots, n\}} > 0\,.$$

This means again that each component is important. We note that the definition is essentially the same as for strategies in Chapter 3, above Corollary 3.1.

The next theorem says that convergence through a balanced weight sequence implies convergence through the whole sequence of integers.

Theorem 4.4. *Let $\boldsymbol{a}_n$ be a balanced weight sequence and c_n a centering sequence, such that $S_{\boldsymbol{a}_n} - c_n \xrightarrow{\mathcal{D}} N(0,1)$. Then $F \in \mathbb{D}(2)$.*

It is important to note that in general the two types of convergence are very different. According to the Corollary in [28] if $F \in \mathbb{D}_{gp}(W)$, for a nondegenerate semistable law W, then there is a balanced weight sequence $\boldsymbol{a}_n$, which contains only two different components, and for which $S_{\boldsymbol{a}_n} - c_n \xrightarrow{\mathcal{D}} W$, where c_n is well determined. However, in this case F is not necessarily contained in the domain of attraction of any stable law. We also note that Megyesi [32] proved for any stable law W that its domain of geometric partial attraction and its domain of attraction coincide. These results show similarity between convergence along a geometric subsequence, and convergence through balanced weight sequence.

There is an interesting problem in connection with such weight sequences: What is the class of infinitely divisible random variables, whose distribution can be obtained as the limit distribution of linear combinations of iid variables with balanced weight sequences. We do not even know whether nonsemistable limits of this type exist or not.

The questions treated here are also interesting in the case of α-stable laws, where $\alpha < 2$. The analogue of Theorem 4.2 remains true, however the validity of Theorems 4.3 and 4.4 is an open problem.

4.2. Proofs

Proof of Theorem 4.1. As we have seen before Theorem 4.1 we may assume that the weights are nonnegative. In this case $F_{k,n}(x) = \mathbf{P}\{a_{k,n}X \leq x\} = F(x/a_{k,n})$. We spell out the conditions again: there is asymptotic normality if and only if

(4.3) $\sum_{k=1}^n \Big[F(-x/a_{k,n}) + 1 - F(x/a_{k,n})\Big] \to 0$ for every $x > 0$, and

(4.4) $\sum_{k=1}^n a_{k,n}^2 \Big\{ \int_{|x|\leq\varepsilon/a_{k,n}} x^2 \,\mathrm{d}F(x) - \Big(\int_{|x|\leq\varepsilon/a_{k,n}} x \,\mathrm{d}F(x)\Big)^2 \Big\} \to 1$,

for every $\varepsilon > 0$.

We may assume, that $\mathbf{E}(X) = 0$. Since $\overline{a}_n \to 0$, each term in (4.4) tends to $\mathrm{Var}(X)$. Thus the validity of (4.4) is equivalent to $\lim_{n\to\infty} \sum_{k=1}^n a_{k,n}^2 = 1/\mathrm{Var}(X)$. Moreover in this case (4.3) also holds. Indeed, $\int_{\mathbb{R}\setminus[-x,x]} y^2 \,\mathrm{d}F(y) \geq x^2(F(-x) + 1 - F(x))$, and since the left side tends to 0 as $x \to \infty$, we have

$$\sum_{k=1}^n \Big[F(-x/a_{k,n}) + 1 - F(x/a_{k,n})\Big] = \sum_{k=1}^n \frac{a_{k,n}^2}{x^2}\, o(1) \to 0\,,$$

where $o(1) \to 0$, as $n \to \infty$, proving (4.3), and thus the statement. ■

Proof of Corollary 4.1. Necessity. According to Theorem 4.1 asymptotic normality implies $\sum_{k=1}^n p_{k,n}^2/a_n^2 \to \mathrm{Var}(X)^{-1}$ and since $\sum_{k=1}^n p_{k,n}^2 \leq \overline{p}_n \leq 1$, we get that c_n is bounded. Therefore $\overline{p}_n/a_n \to 0$ implies $\overline{p}_n \to 0$, and hence $a_n \to 0$ too. Since $a_n \sim \sqrt{\mathrm{Var}(X)\sum_{k=1}^n p_{k,n}^2}$ [for numerical sequences we write $a_n \sim b_n$ if $a_n/b_n \to 1$], we obtain

$$\lim_{n\to\infty} \frac{\overline{p}_n}{\sqrt{\sum_{k=1}^n p_{k,n}^2}} = 0\,,$$

as claimed.

Sufficiency. Put $a_n = \sqrt{\mathrm{Var}(X)\sum_{k=1}^n p_{k,n}^2}$ for the norming sequence. Then $\sum_{k=1}^n p_{k,n}^2/a_n^2 = \mathrm{Var}(X)^{-1}$ and $\overline{a}_n = \overline{p}_n/\sqrt{\mathrm{Var}(X)\sum_{k=1}^n p_{k,n}^2} \to 0$, so by Theorem 4.1 the statement follows. ■

Proof of Theorem 4.2. It is well known that $F \in \mathbb{D}(2)$ if and only if

(4.5) $$\lim_{x\to\infty} \frac{x^2\,[F(-x) + 1 - F(x)]}{\int_{|y|\leq x} y^2 \,\mathrm{d}F(y)} = 0\,.$$

By (4.5) we have

$$\sum_{k=1}^{n} \Big[F(-x/|a_{k,n}|) + 1 - F(x/|a_{k,n}|)\Big] = o(1)\frac{1}{x^2}\sum_{k=1}^{n}\int_{|y|\le x/|a_{k,n}|} a_{k,n}^2\, y^2\,\mathrm{d}F(y)\,,$$

where $o(1)$ is meant as $o(1) \to 0$, if $n \to \infty$. By (4.2) the sum after $o(1)$ on the right-hand side of the equality is bounded for x small enough, and using (4.1) it is easy to see that it is bounded for all $x > 0$. Thus the right-hand side goes to 0. Since the left-hand side converge to $L(x) - R(x)$, where L and R are the Lévy functions as in (4.1), we obtain that both Lévy functions are identically 0, which means that the limit distribution is necessarily normal. ■

Proof of Theorem 4.3. As before we may assume that the weights are nonnegative. Suppose indirectly that $X \notin \mathbb{D}_p(2)$. By the well-known characterization this means that

$$\liminf_{x\to\infty} \frac{x^2\big[F(-x) + 1 - F(x)\big]}{\int_{|y|\le x} y^2\,\mathrm{d}F(y)} > 0\,.$$

Choose $a > 0$, which is smaller than the lim inf above. Hence if x is large enough, we have

$$x^2\Big[F(-x) + 1 - F(x)\Big] > a\int_{|y|\le x} y^2\,\mathrm{d}F(x)\,.$$

Since $\overline{a}_n \to 0$ we obtain

$$\sum_{k=1}^{n} [F(-x/a_{k,n}) + 1 - F(x/a_{k,n})] \ge a\sum_{k=1}^{n}\frac{a_{k,n}^2}{x^2}\int_{|y|\le x/a_{k,n}} y^2\,\mathrm{d}F(y)\,.$$

By (4.3) the left-hand side goes to 0, so the right-hand side also does, which implies by (4.4) that $\sigma = 0$. The contradiction proves the statement. ■

Proof of Theorem 4.4. We assume as before that the weight sequence is nonnegative. If $\mathbf{E}(X^2) < \infty$ then the statement is obvious, therefore we suppose that the variance is infinite. In this case, as we mentioned before, the second term in (4.4) is superfluous. The definition of balancedness implies that there exists $K > 1$, such that $\overline{a}_n/a_{k,n} < K$, for each n and $k = 1, \dots, n$. Then writing x/K instead of x in (4.3) we obtain

$$\sum_{k=1}^{n}\int_{|y|>\frac{1}{Ka_{k,n}}}\mathrm{d}F(y) \ge \sum_{k=1}^{n}\int_{|y|>1/\overline{a}_n}\mathrm{d}F(y) = n\int_{|y|>1/\overline{a}_n}\mathrm{d}F(y)\,,$$

and since the left side tends to 0, so does the right.

Rewriting the left side of (4.4), without the second term, we have for $\varepsilon = 1$

$$\sum_{k=1}^{n} a_{k,n}^2 \int_{|y|<1/a_{k,n}} y^2 \, \mathrm{d}F(y) = \sum_{k=1}^{n} a_{k,n}^2 \int_{|y|<1/\overline{a}_n} y^2 \, \mathrm{d}F(y) + \sum_{k=1}^{n} a_{k,n}^2 \int_{1/a_{k,n} \geq |y| \geq 1/\overline{a}_n} y^2 \, \mathrm{d}F(y) \,,$$

and for the remainder term

$$\sum_{k=1}^{n} a_{k,n}^2 \int_{1/a_{k,n} \geq |y| \geq 1/\overline{a}_n} y^2 \, \mathrm{d}F(y) \leq \sum_{k=1}^{n} \int_{1/a_{k,n} \geq |y| \geq 1/\overline{a}_n} \mathrm{d}F(y) \leq n \int_{|y| \geq 1/\overline{a}_n} \mathrm{d}F(y) \,,$$

which tends to 0. This means that

$$\int_{|y| \leq 1/\overline{a}_n} y^2 \, \mathrm{d}F(y) \sum_{k=1}^{n} a_{k,n}^2 \to 1$$

as $n \to \infty$. Finally, since $n\overline{a}_n^2/K^2 \leq \sum_{k=1}^{n} a_{k,n}^2 \leq n\overline{a}_n^2$, we obtain

$$1 \leq \liminf_{n\to\infty} n\,\overline{a}_n^2 \int_{|y| \leq \frac{1}{\overline{a}_n}} y^2 \, \mathrm{d}F(y) \leq \limsup_{n\to\infty} n\,\overline{a}_n^2 \int_{|y| \leq \frac{1}{\overline{a}_n}} y^2 \, \mathrm{d}F(y) \leq K^2 \,.$$

From this boundedness we show that $F \in \mathbb{D}(2)$, with the same idea as in [23] p. 181. Put $\chi(x) = \int_{|y|>x} \mathrm{d}F(y)$, and $H(x) = \int_{|y|<x} y^2 \, \mathrm{d}F(y)/x^2$. Now $\overline{a}_n \to 0$ implies that for each x large enough we can find $n \in \mathbb{N}$ such that $1/\overline{a}_n < x \leq 1/\overline{a}_{n+1}$. Then clearly $\chi(x) \leq \chi(1/\overline{a}_n)$ and $H(x) \geq H(1/\overline{a}_{n+1}) - \chi(1/\overline{a}_n)$. Thus

$$\frac{\chi(x)}{H(x)} \leq \frac{n\chi(1/\overline{a}_n)}{nH(1/\overline{a}_{n+1}) - n\chi(1/\overline{a}_n)} \,.$$

We have just seen above that $n\chi(1/\overline{a}_n) \to 0$ and $nH(1/\overline{a}_n)$ is bounded, thus $\chi(x)/H(x) \to 0$, as $x \to \infty$, which is exactly the same as (4.5), that is $F \in \mathbb{D}(2)$ as claimed. ■

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Összefoglaló

Legyenek $X_1, X_2, \dots, X_n$ valamely szemistabilis eloszlás geometriai parciális vonzástartományából vett független azonos eloszlású véletlen változók. Ezekre a változókra mint egy szerencsejátékban n játék során nyert nyereményekre (negatív érték esetén veszteségekre) gondolunk. Tegyük fel, hogy Péter, a bankos pontosan egy ilyen játékot játszik n játékos, Pál_1, Pál_2, …, Pál_n mindegyikével, nyereményeik rendre $X_1, X_2, \dots, X_n$. A játékosok, mielőtt még játszanának, nyereményük elosztására előre megállapodhatnak egy $\boldsymbol{p}_n = (p_{1,n}, \dots, p_{n,n})$ osztozkodási stratégiában, melyben a komponensek nemnegatívak és összegük egy. Ennél a stratégiánál Pál_1 kap $p_{1,n}X_1 + p_{2,n}X_2 + \dots + p_{n,n}X_n$ dukátot, Pál_2 kap $p_{n,n}X_1 + p_{1,n}X_2 + \dots + p_{n-1,n}X_n$ dukátot, Pál_3 kap $p_{n-1,n}X_1 + p_{n,n}X_2 + p_{1,n}X_3 + \dots + p_{n-2,n}X_n$ dukátot, …, Pál_n pedig $p_{2,n}X_1 + p_{3,n}X_2 + \dots + p_{n,n}X_{n-1} + p_{1,n}X_n$ dukátot kap. A disszertációban a $p_{1,n}X_1 + p_{2,n}X_2 + \dots + p_{n,n}X_n$ véletlen változó aszimptotikus viselkedését vizsgáljuk, ami esetünkben Pál_1 nyereménye a $\boldsymbol{p}_n$ stratégia mellett.

Az első fejezetben azt a speciális esetet taglaljuk, amikor a szerencsejáték a szentpétervári(p) játék. Péter, a bankos, felajánlja, hogy Pál_1, Pál_2, …, Pál_n játékosok mindegyikével egy-egy általánosított szentpétervári játékot játszik, amelyekben mindegyik Pál $q^{k-1}p$ valószínűséggel nyer r^k dukátot, $k = 1, 2, \dots$, ahol $0 < p < 1$, $q = 1 - p$ és $r = 1/q$. Pál_j nyereményét X_j-vel jelölve, a játékosok megegyeznek, hogy $X_1 + X_2 + \dots + X_n$ össznyereményük önmaguk közötti szétosztására egy $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \dots, p_{n,n})$ valószínűség-eloszlással meghatározott együttműködési stratégiát használnak, ahol tehát $p_{1,n}, p_{2,n}, \dots, p_{n,n} \geq 0$ és $\sum_{j=1}^{n} p_{j,n} = 1$, úgy, hogy Pál_1 $p_{1,n}X_1 + p_{2,n}X_2 + \dots + p_{n,n}X_n$ dukátot, Pál_2 $p_{n,n}X_1 + p_{1,n}X_2 + \dots + p_{n-1,n}X_n$ dukátot, …, Pál_n pedig $p_{2,n}X_1 + p_{3,n}X_2 + \dots + p_{1,n}X_n$ dukátot kap. Végtelen várható értékek összehasonlításával meghatározzuk azokat a stratégiákat, amelyek minden Pál számára eredeti saját nyereményéhez képest extra hozamot eredményeznek annak ellenére, hogy Péter összesen ugyanazt az $X_1 + X_2 + \dots + X_n$ dukátot fizeti ki. Ezek a megengedett stratégiák akkor és csak akkor léteznek, ha q egy speciális egyenletet kielégítő algebrai szám, és ekkor egy megengedett stratégia hozama nem egyéb, mint a stratégia r-alapú logaritmushoz tartozó entrópiájának p/q-szorosa. Megmutatjuk, hogy ez a hozam nemcsak impropriusz Riemann, hanem Lebesgue értelemben is mindig létezik annak ellenére, hogy a klasszikus $p = 1/2$ esettől eltérően az eredeti saját nyereményeket a megengedett stratégiákkal kapott összegek sztochasztikusan csak két játékos esetén dominálják mindig. Legalább három játékos esetén megmutatjuk, hogy a sztochasztikus összehasonlítás általában nem lehetséges. Mint kiderül, ez meg annak ellenére van így, hogy sztochasztikusan domináns helyzetből, tehát például két játékostól indulva egy természetes algoritmussal további játékosokra nyert megengedett stratégiák esetén a sztochasztikus dominancia öröklődik. Sok érdekes speciális esetben meghatározzuk az optimális megengedett stratégiát és ennek maximális hozamát, az általános helyzetre vonatkozóan pedig feltárjuk a kapcsolatos számelméleti természetű nehézségeket.

A második fejezetben még mindig egy speciális esetet tárgyalunk, de már a nyeremények aszimptotikus viselkedésére koncentrálva. Sőt, a speciális eset lehetővé teszi, hogy nemcsak összetartási tételeket, hanem összetartó aszimptotikus sorfejtést is igazoljunk, amire az általános esetben nincs remény. Legyenek tehát $X, X_1, \ldots, X_n$ független általánosított szentpétervári(α, p) változók, azaz melyekre $\mathbf{P}\{X = r^{k/\alpha}\} = q^{k-1}p$, ahol $\alpha \in (0,2)$, $p \in (0,1)$, $q = 1-p$ és $r = 1/q$. Tetszőleges $\boldsymbol{p}_n = (p_{1,n}, \ldots, p_{n,n})$ stratégia esetén definiáljuk az $S_{\boldsymbol{p}_n}^{\alpha,p} = p_{1,n}^{1/\alpha}X_1 + \cdots + p_{n,n}^{1/\alpha}X_n - H_{\alpha,p}(\boldsymbol{p}_n)$ véletlen változót, ahol a $H_{\alpha,p}(\boldsymbol{p}_n)$ csak a stratégiától függő állandó. A fejezet fő eredménye egy aszimptotikus sorfejtés, mely az $S_{\boldsymbol{p}_n}^{\alpha,p}$ és bizonyos $W_{\boldsymbol{p}_n}^{\alpha,p}$ szemistabilis eloszlású véletlen változók összetartási sebességének rendjét határozza meg. Általános esetben nem várhatjuk határeloszlás létezését, hiszen a Doeblin–Gnyegyenko kritérium szerint már az egyenletes stratégia esetén sincs határeloszlás. Azonban abban a speciális esetben, amikor minden komponensre $p_{k,n} = q^{a_{k,n}}$ vagy 0, valamilyen $a_{k,n}$ pozitív egész számra, belátjuk, hogy van határeloszlás, és ekkor összetartó sorfejtéseink hagyományos aszimptotikus sorfejtésekre redukálódnak.

A harmadik fejezetben a problémát teljes általánosságában vizsgáljuk. Legyenek $X_1, X_2, \ldots, X_n$ független, azonos eloszlású véletlen változók, amik benne vannak egy $\alpha \in (0,2)$ kitevős szemistabilis eloszlás geometriai parciális vonzástartományában. Tekintsünk egy tetszőleges $\boldsymbol{p}_n$ osztozkodási stratégiát és definiáljuk a hozzá tartozó $S_{\alpha,\boldsymbol{p}_n} = p_{1,n}^{1/\alpha}X_1/\ell(p_{1,n}) + p_{2,n}^{1/\alpha}X_2/\ell(p_{2,n}) + \cdots + p_{n,n}^{1/\alpha}X_n/\ell(p_{n,n}) - \mu(\boldsymbol{p}_n)$ lineáris kombinációt, ahol $\ell(\cdot)$ egy 0-ban lassú változású függvény, $\mu(\boldsymbol{p}_n)$ pedig valós állandó. A fejezet fő eredménye egy összetartási tétel az $S_{\boldsymbol{p}_n}$ véletlen változó és bizonyos szemistabilis eloszlású $V_{\boldsymbol{p}_n}$ véletlen változó között. Annak ellenére, hogy általában nincs határeloszlás, megadunk olyan –az egyenletes stratégiától nem sokban különböző–

stratégiasorozatot, mely mentén van határeloszlás.

A negyedik fejezetben azt boncolgatjuk, mit mondhatunk akkor, ha a határeloszlás normális. Tetszőleges $\boldsymbol{a}_n = (a_{1,n}, \dots, a_{n,n})$ súlysorozat esetén az $S_{\boldsymbol{a}_n} = a_{1,n}X_1 + \cdots + a_{n,n}X_n$ véletlen változó aszimptotikus normalitását vizsgáljuk a természetes $\lim_{n\to\infty} \max\{|a_{k,n}| : k = 1, \dots, n\} = 0$ elhanyagolhatósági feltétel mellett. A véges szórású esetben szükséges és elegendő feltételt adunk az eloszlásbeli konvergenciára. Megmutatjuk, hogy ha $S_{\boldsymbol{a}_n}$ aszimptotikusan normális egy kiegyensúlyozott $\boldsymbol{a}_n$ súlysorozat mellett, akkor a közös eloszlásfüggvény szükségképpen a normális eloszlás vonzástartományában van.

Független azonos eloszlású véletlen változók lineáris kombinációinak határeloszlása meglepő hasonlóságot mutat részsorozatokon vett eloszlásbeli konvergenciával, és a szemistabilis tulajdonságot is új megvilágításba helyezi.

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