

**Linear combinations of iid random
variables from the domain of
geometric partial attraction of a
semistable law**

Outline of Ph.D. Theses

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1. The generalized n -Paul paradox

This chapter is an extended version of [5]. The enumeration of equations, theorems and corollaries is the same as in the theses, for easier reference.

Peter offers to let Paul toss a possibly biased coin until it lands heads and pays him r^k ducats if this happens on the k^{th} toss, where $r = 1/q$ for $q = 1 - p$ and $p \in (0, 1)$ is the probability of ‘heads’ at each throw. This is the generalized St. Petersburg(p) game, in which $\mathbf{P}\{X = r^k\} = q^{k-1}p$, $k \in \mathbb{N}$, for Paul’s gain X . We assume that Peter plays exactly one such game with each of $n \geq 2$ players, Paul₁, Paul₂, . . . , Paul _{n} , whose independent individual winnings are X_1, X_2, \dots, X_n . The players may agree to use a pooling strategy $\mathbf{p}_n = (p_{1,n}, p_{2,n}, \dots, p_{n,n})$, before they play, where $p_{1,n}, p_{2,n}, \dots, p_{n,n} \geq 0$ and $\sum_{j=1}^n p_{j,n} = 1$. Under this strategy Paul₁ receives $p_{1,n}X_1 + p_{2,n}X_2 + \dots + p_{n,n}X_n$, Paul₂ receives $p_{n,n}X_1 + p_{1,n}X_2 + \dots + p_{n-1,n}X_n$, . . . , and Paul _{n} receives $p_{2,n}X_1 + p_{3,n}X_2 + \dots + p_{n,n}X_{n-1} + p_{1,n}X_n$ ducats. This strategy is fair to every Paul in the sense that their winnings are equally distributed and each receives the same *added value* equal to

$$(1.1) \quad \begin{aligned} A_p(\mathbf{p}_n) &= \mathbf{E}[p_{1,n}X_1 + \dots + p_{n,n}X_n, X_1] \\ &= \int_0^\infty [\mathbf{P}\{p_{1,n}X_1 + \dots + p_{n,n}X_n > x\} - \mathbf{P}\{X_1 > x\}] dx, \end{aligned}$$

whenever the integral is defined. We call a strategy $\mathbf{p}_n = (p_{1,n}, \dots, p_{n,n})$ *admissible* if each of its components is either zero or a nonnegative integer power of q . The entropy of a pooling strategy is $H_r(\mathbf{p}_n) = \sum_{j=1}^n p_{j,n} \log_r 1/p_{j,n}$.

Theorem 1.1. *For any $p \in (0, 1)$ and $n \in \mathbb{N}$, the added value $A_p(\mathbf{p}_n)$ exists as an improper Riemann integral if and only if \mathbf{p}_n is admissible, in which case $A_p(\mathbf{p}_n) = \frac{p}{q} H_r(\mathbf{p}_n)$.*

Csörgő and Simons [4] proved this theorem for the classical St. Petersburg($1/2$) game, played with an unbiased coin. However, in that case they proved the following stronger result: the independent St. Petersburg($1/2$) variables X_1, \dots, X_n can be defined on a rich enough probability space that carries, for each admissible strategy $\mathbf{p}_n = (p_{1,n}, \dots, p_{n,n})$, a St. Petersburg($1/2$) random variable $X_{\mathbf{p}_n}$ and a nonnegative random variable $Y_{\mathbf{p}_n}$ such that $T_{\mathbf{p}_n} = p_{1,n}X_1 + \dots + p_{n,n}X_n = X_{\mathbf{p}_n} + Y_{\mathbf{p}_n}$ almost surely. This implies the stochastic inequality $T_{\mathbf{p}_n} \geq_{\mathcal{D}} X_1$. Hence the integrand in $A_{1/2}(\mathbf{p}_n)$ is nonnegative and thus $A_{1/2}(\mathbf{p}_n)$ is trivially finite as a Lebesgue integral. As the next result shows, stochastic dominance is preserved for two players for an arbitrary St. Petersburg parameter $p \in (0, 1)$.

Theorem 1.2. *For any $p \in (0, 1)$, if $\mathbf{p}_2 = (q^a, q^b)$ is an admissible pooling strategy for some $a, b \in \mathbb{N}$, then $T_{\mathbf{p}_2} = q^a X_1 + q^b X_2$ is stochastically larger than X_1 .*

Surprisingly, for $n \geq 3$ gamblers stochastic dominance generally fails to hold for admissible strategies.

Theorem 1.3. *If $p = (n - 1)/n$, $q = 1/n$ and $n \geq 3$, then neither $S_n = X_1 + \dots + X_n$ nor nX_1 is stochastically larger than the other.*

In view of Theorem 1.2 the integrand in (1.1) is nonnegative whenever \mathbf{p}_2 is admissible, so that the integral $A_p(\mathbf{p}_2)$ described in Theorem 1.1 strengthens to that of a Lebesgue integral when $n = 2$. While the same conclusion holds for $n \geq 3$, Theorem 1.3 rules out so simple a line of reasoning.

Theorem 1.4. *For every parameter $p \in (0, 1)$ and every admissible strategy $\mathbf{p}_n = (p_{1,n}, \dots, p_{n,n})$ the integral $A_p(\mathbf{p}_n)$ in (1.1) is finite as a Lebesgue integral.*

Theorem 1.1 characterizes the pooling strategies that yield added values. However, admissible strategies do *not* exist for all, in fact, for most parameters $p \in (0, 1)$. If for p there exists an admissible strategy, then p is an algebraic number, so the cardinality is at most countable. Nevertheless, it can be shown that there are quite a lot admissible parameters.

Theorem 1.5. *The set of admissible parameters is dense in $(0, 1)$.*

When a given number of our Pauls happen to have admissible strategies, a natural question is: which is the best? In the latter rational case when $p = (m - 1)/m$ for some integer $m \geq 2$, and so $r = 1/q = m \geq 2$ is an integer, the answer is given by the next result. In the theorem $\lfloor y \rfloor$ stands for the lower integer part, $\lceil y \rceil$ for the upper integer part, and $\langle y \rangle$ is the fractional part of y .

Theorem 1.6. *If $p = (r - 1)/r$ and $n = r^{\lfloor \log_r n \rfloor} + (r - 1)r_n$ for some integers $r \geq 2$ and $0 \leq r_n \leq r^{\lfloor \log_r n \rfloor} - 1$, then*

$$A_p(\mathbf{p}_n) = \frac{p}{q} H_r(\mathbf{p}_n) \leq \frac{p}{q} \log_r n - \delta_p(n) =: A_{p,n}^*$$

for every admissible strategy \mathbf{p}_n , where $\delta_p(u) = 1 + (r - 1)\langle \log_r u \rangle - r^{\langle \log_r u \rangle}$, $u > 0$. Moreover, the bound $A_{p,n}^*$ is attainable by means of the admissible strategy

$$\mathbf{p}_n^* = (p_{1,n}^*, \dots, p_{n,n}^*) = (rp_n^*, \dots, rp_n^*, p_n^*, \dots, p_n^*) \quad \text{with} \quad p_n^* = \frac{1}{r^{\lfloor \log_r n \rfloor}},$$

where the number of p_n^* s and rp_n^* s are, respectively,

$$m_{1,p}(n) = \frac{rn - r^{\lceil \log_r n \rceil}}{r - 1} \quad \text{and} \quad m_{2,p}(n) = \frac{r^{\lceil \log_r n \rceil} - n}{r - 1}.$$

Apart from reorderings of the components of \mathbf{p}_n^* , the point of maximum is unique.

Finally, we show a strategy generating branching algorithm, which has an interesting property concerning stochastic domination. For any admissible parameter $p \in (0, 1)$, let $(q^{a_1}, q^{a_2}, \dots, q^{a_n})$ and $(q^{b_1}, q^{b_2}, \dots, q^{b_m})$ be admissible strategies for n and m Pauls for any $n, m \geq 2$. Substituting $q^{a_k+b_1} + q^{a_k+b_2} + \dots + q^{a_k+b_m} = q^{a_k}$ for q^{a_k} , where $k \in \{1, \dots, n\}$ is arbitrary, we obtain a strategy $(q^{d_1}, q^{d_2}, \dots, q^{d_{n+m-1}})$ for $n + m - 1$ gamblers, where the sequence $d_1 \geq d_2 \geq \dots \geq d_{n+m-1}$ is a nonincreasing rearrangement of the sequence $a_1, \dots, a_{k-1}, a_k + b_1, \dots, a_k + b_m, a_{k+1}, \dots, a_n$. We say that a strategy $\mathbf{p}_n = (p_{1,n}, \dots, p_{n,n})$ is *stochastically dominant* if $p_{1,n}X_1 + \dots + p_{n,n}X_n \geq_{\mathcal{D}} X_1$. The last theorem states that the branching algorithm preserves stochastic dominance. Choosing first $n = m = 2$, it may be used in conjunction with Theorem 1.2 as a starting point.

Theorem 1.7. *If the strategies $(q^{a_1}, q^{a_2}, \dots, q^{a_n})$ and $(q^{b_1}, q^{b_2}, \dots, q^{b_m})$ are both stochastically dominant, then the generated strategy $(q^{d_1}, q^{d_2}, \dots, q^{d_{n+m-1}})$ is also stochastically dominant.*

2. Merging asymptotic expansions for generalized St. Petersburg games

The results of this chapter are from [2].

We further generalize the St. Petersburg game. Peter offers to let Paul toss a possibly biased coin repeatedly until it lands heads and pays him $r^{k/\alpha}$ ducats if this happens on the k^{th} toss, where $r = 1/q$ for $q = 1 - p$, and $p \in (0, 1)$ is the probability of heads on each throw, while $\alpha \in (0, 2)$ is a payoff parameter.

For the bias parameter $p \in (0, 1)$, the payoff or tail parameter $\alpha \in (0, 2)$ and a third parameter $\gamma \in (q, 1]$, consider the infinitely divisible random variable

$$(2.2) \quad W_\gamma^{\alpha,p} = \frac{1}{\gamma^{1/\alpha}} \left\{ \sum_{m=0}^{-\infty} r^{m/\alpha} \left[Y_m^{p,\gamma} - \frac{p\gamma}{qr^m} \right] + \sum_{m=1}^{\infty} r^{m/\alpha} Y_m^{p,\gamma} \right\} + S_\gamma^{\alpha,p},$$

where $\dots, Y_{-2}^{p,\gamma}, Y_{-1}^{p,\gamma}, Y_0^{p,\gamma}, Y_1^{p,\gamma}, Y_2^{p,\gamma}, \dots$ are independent random variables such that $Y_m^{p,\gamma}$ has the Poisson distribution with mean $pr\gamma q^m = p\gamma/(qr^m)$, and where $s_\gamma^{\alpha,p}$ is constant. Let $G_{\alpha,p,\gamma}(x) = \mathbf{P}\{W_\gamma^{\alpha,p} \leq x\}$ denote its distribution and

$$(2.3) \quad \mathbf{g}_{\alpha,p,\gamma}(t) = \mathbf{E}(e^{itW_\gamma^{\alpha,p}}) = \int_{-\infty}^{\infty} e^{itx} dG_{\alpha,p,\gamma}(x) = e^{y_\gamma^{\alpha,p}(t)}, \quad t \in \mathbb{R},$$

its characteristic function, where

$$y_\gamma^{\alpha,p}(t) = it s_\gamma^{\alpha,p} + \sum_{l=0}^{-\infty} \left(\exp\left\{ \frac{itr \frac{l}{\alpha}}{\gamma^{\frac{l}{\alpha}}} \right\} - 1 - \frac{itr \frac{l}{\alpha}}{\gamma^{\frac{l}{\alpha}}} \right) \frac{p\gamma}{qr^l} + \sum_{l=1}^{\infty} \left(\exp\left\{ \frac{itr \frac{l}{\alpha}}{\gamma^{\frac{l}{\alpha}}} \right\} - 1 \right) \frac{p\gamma}{qr^l}.$$

The form of the exponent of the characteristic function immediately implies that for every $p \in (0, 1)$ and $\gamma \in (q, 1]$ the infinitely divisible distribution of $W_\gamma^{\alpha,p}$ is semistable with exponent α . It follows that $G_{\alpha,p,\gamma}(\cdot)$ is infinitely many times differentiable.

Consider a sequence of pooling strategies $\{\mathbf{p}_n = (p_{1,n}, \dots, p_{n,n})\}_{n=1}^{\infty}$, and assume that $\bar{p}_n = \max\{p_{1,n}, \dots, p_{n,n}\} \rightarrow 0$. Our first interest in this chapter is the asymptotic distribution of

$$(2.8) \quad S_{\mathbf{p}_n}^{\alpha,p} = \sum_{k=1}^n p_{k,n}^{1/\alpha} X_k - \frac{p}{q} H_{\alpha,p}(\mathbf{p}_n),$$

a particular type of linear combinations when $\alpha \neq 1$, where $H_{\alpha,p}(\mathbf{p}_n)$ is a constant depending on the strategy. Even though $p_{1,n}^{1/\alpha}, \dots, p_{n,n}^{1/\alpha}$ sum to one, and hence form a strategy only for $\alpha = 1$, it is a major technical step to come up with a merging approximation in terms of the distribution functions of the semistable random variables

$$(2.9) \quad W_{\mathbf{p}_n}^{\alpha,p} = \begin{cases} \sum_{k=1}^n p_{k,n}^{1/\alpha} W_{1,k}^{\alpha,p}, & \text{if } \alpha \neq 1, \\ \sum_{k=1}^n p_{k,n} W_{1,k}^{1,p} - \frac{p}{q} H_{1,p}(\mathbf{p}_n), & \text{if } \alpha = 1, \end{cases}$$

where the random variables $W_{1,1}^{\alpha,p}, W_{1,2}^{\alpha,p}, \dots, W_{1,n}^{\alpha,p}$ are independent copies of $W_1^{\alpha,p}$, given by substituting $\gamma = 1$ in (2.2). The characteristic and the distribution functions will be denoted by $\mathbf{g}_{\alpha,p,\mathbf{p}_n}(t) = \mathbf{E}(e^{itW_{\mathbf{p}_n}^{\alpha,p}})$ and $G_{\alpha,p,\mathbf{p}_n}(x) = \mathbf{P}\{W_{\mathbf{p}_n}^{\alpha,p} \leq x\}$ respectively. It is easy to see that $W_{\mathbf{p}_n}^{\alpha,p}$ is indeed a semistable random variable with exponent α for an arbitrary strategy \mathbf{p}_n .

Fix any strategy $\mathbf{p}_n = (p_{1,n}, \dots, p_{n,n})$, and consider the position parameters $\gamma_{k,n} = 1/(p_{k,n} r^{\lceil \log_r 1/p_{k,n} \rceil}) \in (q, 1]$ for each component $k = 1, 2, \dots, n$ for which $p_{k,n} > 0$. Roughly speaking $\gamma_{k,n} \in (q, 1]$ determines the position

of $p_{k,n}$ between two consecutive powers of r . Recalling formula (2.3) for the ingredients and the notation $\mathbf{g}_{\alpha,p,\mathbf{p}_n}(t) = \mathbf{E}(e^{itW_{\mathbf{p}_n}^{\alpha,p}})$ at (2.9), for $t \in \mathbb{R}$ we introduce the complex-valued function $\mathbf{g}_{\mathbf{p}_n}^{\alpha,p}(t)$, defined for $\alpha \neq 1$ as

$$\begin{aligned} \mathbf{g}_{\mathbf{p}_n}^{\alpha,p}(t) = \mathbf{g}_{\alpha,p,\mathbf{p}_n}(t) & \left[1 - \frac{1}{2} \sum_{k=1}^n p_{k,n}^2 [y_{\gamma_{k,n}}^{\alpha,p}(t)]^2 + it s_1^{\alpha,p} \sum_{k=1}^n p_{k,n}^{1+\frac{1}{\alpha}} y_{\gamma_{k,n}}^{\alpha,p}(t) \right. \\ & \left. + \frac{t^2}{2} \left\{ (s_1^{\alpha,p})^2 + \frac{p}{q - q^{2/\alpha}} \right\} \sum_{k=1}^n p_{k,n}^{\frac{2}{\alpha}} \right], \end{aligned}$$

where the constant $s_1^{\alpha,p} = p/(q - q^{1/\alpha})$ is from (2.2), and for $\alpha = 1$ as

$$\begin{aligned} \mathbf{g}_{\mathbf{p}_n}^{1,p}(t) = \mathbf{g}_{1,p,\mathbf{p}_n}(t) & \left[1 - \frac{1}{2} \sum_{k=1}^n p_{k,n}^2 [y_{\gamma_{k,n}}^{1,p}(t)]^2 - it \frac{p}{q} \sum_{k=1}^n p_{k,n}^2 y_{\gamma_{k,n}}^{1,p}(t) \log_r \frac{1}{p_{k,n}} \right. \\ & \left. + \frac{t^2}{2} \left\{ \frac{p^2}{q^2} \sum_{k=1}^n p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}} + \frac{1}{q} \sum_{k=1}^n p_{k,n}^2 \right\} \right]. \end{aligned}$$

Consider finally the function $G_{\mathbf{p}_n}^{\alpha,p}(\cdot)$ on \mathbb{R} that has Fourier–Stieltjes transform $\mathbf{g}_{\mathbf{p}_n}^{\alpha,p}(t)$, that is,

$$(2.15) \quad \mathbf{g}_{\mathbf{p}_n}^{\alpha,p}(t) = \int_{-\infty}^{\infty} e^{itx} dG_{\mathbf{p}_n}^{\alpha,p}(x), \quad t \in \mathbb{R}.$$

The main result for the merging approximation of the distribution function of $S_{\mathbf{p}_n}^{\alpha,p}$ from (2.8) is the following

Theorem 2.1. *For any sequence of strategies $\{\mathbf{p}_n = (p_{1,n}, \dots, p_{n,n})\}_{n \in \mathbb{N}}$,*

$$\sup_{x \in \mathbb{R}} \left| \mathbf{P}\{S_{\mathbf{p}_n}^{\alpha,p} \leq x\} - G_{\mathbf{p}_n}^{\alpha,p}(x) \right| = \begin{cases} O(\bar{p}_n^2), & \text{if } 0 < \alpha < 1/2, \\ O(\bar{p}_n^{1/\alpha}), & \text{if } 1/2 \leq \alpha < 3/2; \\ O(\bar{p}_n^{(4-2\alpha)/\alpha}), & \text{if } 3/2 \leq \alpha < 2, \end{cases}$$

where $\bar{p}_n = \max\{p_{1,n}, \dots, p_{n,n}\}$.

For the uniform strategy $\mathbf{p}_n^\diamond = (1/n, \dots, 1/n)$ our theorem reduces to Proposition in [1], where a correction term was missed. That the inclusion of that term would be a desirable adjustment in [1], at least for $\alpha \neq 1$, was noticed by Pap [8]. Hence for any strategy \mathbf{p}_n the definition of $G_{\mathbf{p}_n}^{\alpha,p}(\cdot)$ in (2.15) is a suitable generalization of the desired full form $G_{\mathbf{p}_n^\diamond}^{\alpha,p}(\cdot)$.

As noted between (2.8) and (2.9), the sum of the weights $p_{1,n}^{1/\alpha}, \dots, p_{n,n}^{1/\alpha}$ in $S_{\mathbf{p}_n}^{\alpha,p}$ adds to unity only if $\alpha = 1$, so for $\alpha \neq 1$ they cannot represent a pooling

strategy. However with a transformation we can easily rewrite Theorem 2.1 in an equivalent, more natural form.

For admissible strategies all $p_{k,n}$ nonzero members of \mathbf{p}_n is an integer power of q , thus the corresponding $\gamma_{k,n} = 1$. Using the definition of $\mathbf{g}_{\mathbf{p}_n}^{\alpha,p}$, we see that in the admissible case there exists a proper limiting distribution, and moreover we have real asymptotic expansions attached to this asymptotic distribution. Concentrating on the dominant terms in Theorem 2.1, we obtain the following

Corollary 2.2. *For any sequence $\{\mathbf{p}_n = (p_{1,n}, \dots, p_{n,n})\}_{n \in \mathbb{N}}$ of admissible strategies, for $\alpha \in (0, 1)$,*

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left| \mathbf{P}\{S_{\mathbf{p}_n}^{\alpha,p} \leq x\} - \left[G_{\alpha,p,1}(x) - G_{\alpha,p,1}^{(0,2)}(x) \frac{1}{2} \sum_{k=1}^n p_{k,n}^2 \right] \right| \\ = \begin{cases} O(\bar{p}_n^2), & \text{if } 0 < \alpha \leq 1/2, \\ O(\bar{p}_n^{1/\alpha}), & \text{if } 1/2 < \alpha < 1; \end{cases} \end{aligned}$$

for $\alpha = 1$,

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left| \mathbf{P}\{S_{\mathbf{p}_n}^{1,p} \leq x\} - \left[G_{1,p,1}(x) + G_{1,p,1}^{(1,1)}(x) \frac{p}{q} \sum_{k=1}^n p_{k,n}^2 \log_r \frac{1}{p_{k,n}} \right. \right. \\ \left. \left. - G_{1,p,1}^{(2,0)}(x) \frac{p^2}{2q^2} \sum_{k=1}^n p_{k,n}^2 \log_r^2 \frac{1}{p_{k,n}} \right] \right| = O(\bar{p}_n); \end{aligned}$$

and for $\alpha \in (1, 2)$,

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left| \mathbf{P}\{S_{\mathbf{p}_n}^{\alpha,p} \leq x\} - \left[G_{\alpha,p,1}(x) - G_{\alpha,p,1}^{(2,0)}(x) \left\{ \frac{p^2}{(q - q^{1/\alpha})^2} + \frac{p}{q - q^{2/\alpha}} \right\} \frac{1}{2} \sum_{k=1}^n p_{k,n}^{2/\alpha} \right] \right| \\ = \begin{cases} O(\bar{p}_n^{1/\alpha}), & \text{if } 1 < \alpha \leq 3/2, \\ O(\bar{p}_n^{(4-2\alpha)/\alpha}), & \text{if } 3/2 < \alpha < 2. \end{cases} \end{aligned}$$

3. Merging of linear combinations to semi-stable laws

This chapter is based on the paper [6].

Consider the semistable random variable $W(\psi_1^\alpha, \psi_2^\alpha, 0)$, that has characteristic function

$$\mathbf{E}(e^{itW(\psi_1^\alpha, \psi_2^\alpha, 0)}) = \exp\left\{-\frac{\sigma^2}{2}t^2 + \int_0^\infty \beta_t(\psi_1^\alpha(u)) du + \int_0^\infty \beta_t(-\psi_2^\alpha(u)) du\right\},$$

where $\beta_t(x) = e^{itx} - 1 - \frac{itx}{1+x^2}$ and

$$(3.3) \quad \psi_j^\alpha(s) = -\frac{M_j(s)}{s^{1/\alpha}}, \quad s > 0, \quad j = 1, 2,$$

where M_1 and M_2 are non-negative, right-continuous functions on $(0, \infty)$, either identically zero or bounded away from both zero and infinity, such that at least one of them is not identically zero, the functions $\psi_j^\alpha(\cdot)$ are non-decreasing and the multiplicative periodicity property $M_j(cs) = M_j(s)$ holds for all $s > 0$, for some constant $c > 1$, $j = 1, 2$. Megyesi [7] showed that every semistable distribution up to a constant can be represented in this way. We also introduce $V(\psi_1^\alpha, \psi_2^\alpha, 0) = W(\psi_1^\alpha, \psi_2^\alpha, 0) + \theta(\psi_1^\alpha) - \theta(\psi_2^\alpha)$, where $\theta(\psi)$ is a constant, and for its distribution function we put $G_{\psi_1^\alpha, \psi_2^\alpha, 0}(x) = \mathbf{P}\{V(\psi_1^\alpha, \psi_2^\alpha, 0) \leq x\}$.

Let the distribution function F be in the domain of geometric partial attraction of the semistable law $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$, and let X_1, X_2, \dots be independent random variables with the common distribution function $F(\cdot)$. We consider a sequence of strategies $\{\mathbf{p}_n\}$ that satisfies the asymptotic negligibility condition $\bar{p}_n = \max\{p_{j,n} : j = 1, 2, \dots, n\} \rightarrow 0$. Our main interest in this chapter is the asymptotic distribution of the random variable

$$(3.8) \quad S_{\alpha, \mathbf{p}_n} = \sum_{j=1}^n \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} X_j - \sum_{j=1}^n \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} \int_{p_{j,n}}^{1-p_{j,n}} Q(s) ds,$$

where the slowly varying function $l(\cdot)$ is from the representation of the quantile function Q corresponding to F .

For $\lambda > 0$, define $\lambda\psi(s) = \psi(s/\lambda)$ and put $\psi_j^{\alpha, \lambda}(s) = \lambda^{-1/\alpha} \lambda\psi_j^\alpha(s) = -M_j(s/\lambda)s^{-1/\alpha}$, $s > 0$, where the functions M_j are from (3.3), $j = 1, 2$. Introduce

$$(3.9) \quad V_{\alpha, \lambda}(M_1, M_2) = V(\psi_1^{\alpha, \lambda}, \psi_2^{\alpha, \lambda}, 0) \text{ and } \mathbf{E}(e^{itV_{\alpha, \lambda}(M_1, M_2)}) = e^{y_{\alpha, \lambda}(t)}, \quad t \in \mathbb{R},$$

and notice the identity $V_{\alpha,\lambda}(M_1, M_2) = \lambda^{-1/\alpha} V(\lambda\psi_1^\alpha, \lambda\psi_2^\alpha, 0)$. For simplicity put $\gamma_{j,n} = \gamma_{1/p_{j,n}}$ if $p_{j,n} > 0$, $j = 1, \dots, n$, where γ_x is a similar positional parameter, as in the previous chapter. The merging semistable approximation to the distribution functions of S_{α,\mathbf{p}_n} in (3.8) is given in the following main result by the distribution functions $G_{\alpha,\mathbf{p}_n}(x) = \mathbf{P}\{V_{\alpha,\mathbf{p}_n} \leq x\}$, $x \in \mathbb{R}$, of random variables V_{α,\mathbf{p}_n} that have characteristic functions

$$(3.11) \quad \mathbf{E}(e^{itV_{\alpha,\mathbf{p}_n}}) = \int_{-\infty}^{\infty} e^{itx} dG_{\alpha,\mathbf{p}_n}(x) = \exp\left\{\sum_{j=1}^n p_{j,n} y_{\alpha,\gamma_{j,n}}(t)\right\}, \quad t \in \mathbb{R},$$

where $y_{\alpha,\gamma_{j,n}}(\cdot)$ is the exponent function in the characteristic function of $V_{\alpha,\gamma_{j,n}}$ in (3.9).

Theorem 3.1. *For any sequence $\{\mathbf{p}_n\}_{n=1}^{\infty}$ of strategies such that $\bar{p}_n \rightarrow 0$,*

$$\sup_{x \in \mathbb{R}} |\mathbf{P}\{S_{\alpha,\mathbf{p}_n} \leq x\} - G_{\alpha,\mathbf{p}_n}(x)| \rightarrow 0.$$

It follows from the formula (3.11) that for the uniform strategies $\mathbf{p}_n^\diamond = (1/n, 1/n, \dots, 1/n)$ the distributional equality $V_{\alpha,\mathbf{p}_n} \stackrel{\mathcal{D}}{=} V_{\alpha,\gamma_n}(M_1, M_2)$ holds, and hence Theorem 3.1 reduces to the most important special case of full sums in Theorem 2 in [3].

As noted before, there is real pooling of winnings only if $\alpha = 1$ and $l(\cdot) \equiv 1$ when the sum of the coefficients in (3.8) is 1. However, by a transformation, similarly as in the St. Petersburg case, we obtain a version of Theorem 3.1 that is satisfactory in this respect.

Now we show that for special sequences $\{\mathbf{p}_n\}$ the merge in Theorem 3.1 reduces to ordinary limit theorems. We call a sequence $\{\mathbf{p}_n\}_{n=1}^{\infty}$ of strategies *balanced* if

$$\liminf_{n \rightarrow \infty} \frac{\min\{p_{j,n} : j = 1, 2, \dots, n\}}{\max\{p_{j,n} : j = 1, 2, \dots, n\}} > 0.$$

Roughly speaking this condition means that each component is important.

Classical theory says that if a limiting distribution exists for the uniform strategies $\mathbf{p}_n^\diamond = (1/n, 1/n, \dots, 1/n)$, it must be stable. As an essence of semistability, the following corollary claims that semistable limiting distributions can be achieved by such balanced strategies that practically consist of only two different components.

Corollary 3.1. *For an arbitrary $\kappa \in (\mathbf{c}^{-1}, 1]$ there exists a balanced sequence $\{\mathbf{p}_n\}_{n=1}^{\infty}$ of strategies such that $S_{\alpha,\mathbf{p}_n} \xrightarrow{\mathcal{D}} V_{\alpha,\kappa}(M_1, M_2)$, where the random variable $V_{\alpha,\kappa}(M_1, M_2)$ is defined in (3.9). Moreover, for each $n \in \{2, 3, \dots\}$*

the strategy $\mathbf{p}_n = (p_{1,n}, p_{2,n}, \dots, p_{n,n})$ can be constructed in such a way that there are at most two different values among its first $n - 1$ components.

For the proof of these results we need to work out the general theory of merge. We say that the random variables X_n and Y_n , or their distribution functions F_n and G_n , merge together if $L(F_n, G_n) \rightarrow 0$, where $L(\cdot, \cdot)$ stands for the Lévy-metric.

Theorem 3.3. *If $\{G_n\}_{n=1}^\infty$ is stochastically compact, then $L(F_n, G_n) \rightarrow 0$ if and only if $\phi_n(t) - \psi_n(t) \rightarrow 0$ for every $t \in \mathbb{R}$, where ϕ_n and ψ_n are the corresponding characteristic functions.*

The next theorem is the basic tool in the proof of Theorem 3.1. It says that if G_n is absolutely continuous for all $n \in \mathbb{N}$ and the corresponding density functions are uniformly bounded, then even uniform convergence holds under the same conditions.

Theorem 3.4. *Assume that $\{G_n\}_{n=1}^\infty$ is stochastically compact and there is a constant $K > 0$ such that $\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}} |G'_n(x)| \leq K$. Then $F_n(x) - G_n(x) \rightarrow 0$ at every $x \in \mathbb{R}$ if and only if $\phi_n(t) - \psi_n(t) \rightarrow 0$ at every $t \in \mathbb{R}$. Moreover, if this holds, then in fact the convergence is uniform, so that $\sup_{x \in \mathbb{R}} |F_n(x) - G_n(x)| \rightarrow 0$.*

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