

THESIS OF PH.D. DISSERTATION

MONOIDAL INTERVALS

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1. INTRODUCTION

The story of monoidal intervals does not have a long history, but it is related to a central theme in universal algebra: composition of operations. Sets of operations that are closed under composition naturally arise in logic (e.g., in the propositional calculus of two-valued or many-valued logics), in algebra (e.g., in studying word problems), and in computer science (e.g., in the synthesis of automata). E. L. Post [Pos41] started to investigate composition-closed sets of operations on a 2-element set (that is, composition-closed sets of truth functions) in order to understand all possible propositional calculi in 2-valued logic. P. Hall [Hal58] was lead to the concept of a **clone**, which can be defined as a composition-closed set of operations containing all projection operations, by studying the word problem for various classes of groups.

For each set A , the clones on A form a complete lattice \mathbb{CL}_A with respect to set-theoretic inclusion. Post's result mentioned in the preceding paragraph is a complete description of all members of the clone lattice $\mathbb{CL}_{\{0,1\}}$. It turns out that if A has two elements, then there are \aleph_0 clones on A . The situation changes dramatically when A has more than two elements. In [JM59] Ju. I. Janov and A. A. Mučnik proved that on a finite set A with more than two elements there are 2^{\aleph_0} clones. Moreover, as the following results of A. A. Bulatov show, the structure of the clone lattice is rather complicated; namely, for any finite set A ,

- if $|A| \geq 3$ then the subsemigroup lattice of the absolutely free one-generated semigroup can be embedded into the clone lattice \mathbb{CL}_A ([Bul92]);
- if $|A| \geq 4$ then any direct product of countably many finite lattices can be embedded into the clone lattice \mathbb{CL}_A ([Bul94]).

Next we explain how the study of monoidal intervals may help understanding the structure of the clone lattice.

Let A be a set. For arbitrary clone \mathcal{C} on A the set of unary operations in \mathcal{C} is clearly a transformation monoid on A . Furthermore, it is not hard to show (see Á. Szendrei [Sze86], Proposition 3.1) that for arbitrary transformation monoid M on A the clones in which the set of unary operations is M form an interval $\text{Int}(M)$ in the clone lattice \mathbb{CL}_A . Such an interval is called a **monoidal interval**. If A is finite, then there are only finitely many transformation monoids on A . Hence the monoidal intervals $\text{Int}(M)$ partition the clone lattice \mathbb{CL}_A into finitely many blocks. Since \mathbb{CL}_A has cardinality 2^{\aleph_0} if $|A| \geq 3$, one might expect that 'for most M ' the monoidal interval $\text{Int}(M)$ contains uncountably many clones. This expectation is justified by the fact (cf. [Dor]) that if $|A| = 3$, then more

than half of the monoidal intervals have cardinality 2^{\aleph_0} . Nevertheless, it turns out that for many interesting transformation monoids M the interval $\text{Int}(M)$ is countable. So, studying these intervals may lead to a better understanding of some parts of the clone lattice \mathbb{CL}_A .

The monoidal intervals are also related to the following unsolved problem on the congruences of the clone lattice: If A is a finite set with more than two elements, does \mathbb{CL}_A have a nontrivial congruence? The relationship is revealed by a result of A. A. Krokhin [Kro01b] proving that any proper congruence of \mathbb{CL}_A is a subrelation of the equivalence relation whose equivalence classes are the monoidal intervals. We note that for the case when A has only two elements, the congruences of \mathbb{CL}_A have been determined by Krokhin–Semigrodskikh [KS01], using Post’s description of \mathbb{CL}_A .

The problem of classifying all monoids on a finite set A according to the cardinalities of the corresponding monoidal intervals was first raised by Á. Szendrei [Sze86]. For the case when A is a two-element set Post’s description of the clone lattice provides a complete solution to this problem: there are three finite and three infinite intervals. For the case when A is a finite set with more than two elements, and hence the clone lattice has cardinality 2^{\aleph_0} , I. G. Rosenberg and N. Sauer in [RS] observed that each monoidal interval in \mathbb{CL}_A either has cardinality 2^{\aleph_0} or is countable (see also M. Pinsker [Pin08]). Thus, Szendrei’s problem can be refined as follows (see A. A. Krokhin [Kro97b]): for which transformation monoids does the corresponding monoidal interval have cardinality

- 1,
- finite but greater than 1,
- \aleph_0 ,
- 2^{\aleph_0} ?

We conclude this section with an overview of some known results related to this problem and a summary of our contributions. The dissertation is based on the articles [Dor02], [Dor07], and [Dor08].

Collapsing monoids

A monoid M on A is called **collapsing** if the interval $\text{Int}(M)$ has only one element, namely the essentially unary clone generated by M .

The first result exhibiting a large family of collapsing monoids is due to P. P. Pálffy [Pal84]; soon after its discovery the result became influential in development of the structure theory of finite algebras called ‘tame congruence theory’. Pálffy’s theorem states that if M is a transformation monoid on a finite set A with more than two elements such that M contains all constant transformations and

each nonconstant member of M is a permutation, then M is collapsing unless M is the monoid of unary polynomial operations of a vector space.

In Theorem 12 we generalize this theorem.

Despite the fact that ‘for most M ’ the monoidal interval $\text{Int}(M)$ is expected to contain uncountably many clones, in Theorem 3 we proved that there are large intervals in the submonoid lattice of the full transformation monoid such that all members of these intervals are collapsing.

For permutation groups the results known so far indicate that ‘large’ permutation groups, e.g. all primitive permutation groups, are collapsing (cf. P. P. Pálffy and Á. Szendrei [PSz82] and K. A. Kearnes and Á. Szendrei [KSz01]). This motivated us in extending the investigation of collapsing monoids to ‘large’ inverse monoids. We investigate the monoidal intervals $\text{Int}(M)$ where M belongs to a class of inverse transformation monoids constructed from finite lattices. These inverse monoids arise from finite lattices by applying the construction introduced by T. Saito and M. Katsura in [SK92] to describe maximal inverse transformation monoids. We describe a necessary and sufficient condition for an inverse monoid constructed from a finite lattice to be collapsing (Theorem 6).

Finite monoidal intervals with more than one element

The earliest result concerning monoidal intervals was the description of the monoidal interval that corresponds to the full transformation monoid $T(A)$: this interval is an $(|A| + 1)$ -element chain (G. A. Burle [Bur67]).

In Pálffy’s theorem, if M coincides with the monoid of all unary polynomial operations of a finite vector space over a finite field then $\text{Int}(M)$ is a 2-element chain (cf. P. P. Pálffy [Pal84] and Theorem 1).

Our result in Theorem 14 is closely related to the latter statement.

Infinite monoidal intervals

We discussed earlier that \aleph_0 and 2^{\aleph_0} are the only possibilities for the cardinality of an infinite monoidal interval if the base set A is finite. As for \aleph_0 , the first, and so far the only construction of a transformation monoid M with $|\text{Int}(M)| = \aleph_0$ is due to A. A. Krokhin in [Kro97b].

In contrast, we know a lot of examples of transformation monoids M for which $|\text{Int}(M)| = 2^{\aleph_0}$ holds. For example, the monoidal interval corresponding to the one-element monoid (consisting of the identity operation only) satisfies this condition.

We present a family of inverse transformation monoids constructed from finite lattices using the Saito–Katsura construction in [SK92], for which the corresponding monoidal intervals have cardinality 2^{\aleph_0} (Theorem 9 and Theorem 8).

2. PRELIMINARIES

This section is devoted to a survey of the basic concepts and techniques that will be used. In this chapter we discuss clones on finite sets and their properties, and monoidal intervals.

As usual, for a set X the set of all subsets of X will be denoted by $P(X)$. Let X, Y, Y' , and Z be sets for which $Y \subseteq Y'$ holds. By the composition of the maps $\alpha: X \rightarrow Y$ and $\beta: Y' \rightarrow Z$ we will mean the map $X \rightarrow Z$, $x \mapsto (x\alpha)\beta$, denoted by $\alpha \circ \beta$. For arbitrary subset W of X the restriction of the map α to the set W is the map $\alpha|_W: W \rightarrow \alpha(W)$, $x \mapsto \alpha(x)$.

For a finite set A we will denote the full transformation semigroup, the symmetric group, and the set of unary constant operations on A by $T(A)$, $S(A)$, and $C(A)$, respectively. For an arbitrary element a of A we will use the notation c_a for the unary constant operation on A with value a .

For the set of positive integers we will use the notation \mathbb{N} , and we will refer to them as natural numbers.

Clones

Let A be a set and n be a positive integer. An **n -ary operation on A** is a function $f: A^n \rightarrow A$. An operation is called **finitary** if it is n -ary for a positive integer n . The set of all finitary operations on A will be denoted by \mathcal{O}_A . The **superposition** of an n -ary operation $f \in \mathcal{O}_A$ by a k -ary operations $g_1, \dots, g_n \in \mathcal{O}_A$ is the k -ary operation $f(g_1, \dots, g_n) \in \mathcal{O}_A$ defined by the rule

$$f(g_1, \dots, g_n)(x_1, \dots, x_k) = f(g_1(x_1, \dots, x_k), \dots, g_n(x_1, \dots, x_k)),$$

and for positive integers n and $i \leq n$ the **i -th n -ary projection** is the operation

$$e_i^{(n)}: A^n \rightarrow A, e_i^{(n)}(x_1, \dots, x_n) \mapsto x_i.$$

A set \mathcal{C} of finitary operations on a set A is said to be a **clone** if it contains all the projections and is closed under superposition of operations. It is obvious that \mathcal{O}_A and the set \mathcal{P}_A of all projections on A are clones. Since the intersection of an arbitrary family of clones on A is also a clone, the set of all clones on A constitutes a complete lattice with respect to the set-theoretic inclusion. This lattice will be denoted by \mathbb{CL}_A . The greatest and the least elements of \mathbb{CL}_A are \mathcal{O}_A and \mathcal{P}_A , respectively. Furthermore, we can define the **clone generated by a subset F** of \mathcal{O}_A as the intersection of all clones that contain F . This clone will be denoted by $\langle F \rangle$. It is easy to see that $\langle F \rangle$ is the least clone containing

F . If F is a finite subset of \mathcal{O}_A , say $F = \{f_1, \dots, f_s\}$, then we write $\langle f_1, \dots, f_s \rangle$ instead of $\langle \{f_1, \dots, f_s\} \rangle$. For a positive integer n , the set of all n -ary operations of in a clone \mathcal{C} will be denoted by $\mathcal{C}^{(n)}$.

For an algebra $\mathbb{A} = (A; F)$ there is a clone that can be naturally attached to it, the **clone of term operations** of \mathbb{A} , which is the clone generated by its fundamental operations F . This clone will be denoted by $\text{Clo}(\mathbb{A})$. The algebras \mathbb{A} and \mathbb{B} with the same universe are said to be **term equivalent** if their clones of term operations coincide, i.e., $\text{Clo}(\mathbb{A}) = \text{Clo}(\mathbb{B})$. It is worth mentioning that term equivalent algebras have the same subalgebras and congruences. It is easy to see that every clone on A can be obtained as a clone of term operations of a suitable algebra with universe A .

An n -ary operation $f \in \mathcal{O}_A$ is said to **depend on its i -th variable** ($1 \leq i \leq n$) if there are elements $a_1, \dots, a_{i-1}, a_i, a'_i, a_{i+1}, \dots, a_n$ of A such that

$$f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n).$$

Otherwise the i -th variable of f is called **fictitious**. We call the operation f **essentially k -ary** if it depends on exactly k variables of its all variables.

For a natural number k a **k -ary relation on A** is a subset of A^k . A relation is finitary if it is k -ary for a positive integer k . We will denote by \mathcal{R}_A the set of all finitary relations on A . Let m and n be positive integers, and let $\varrho \in \mathcal{R}_A$ be an m -ary relation and $f \in \mathcal{O}_A$ be an n -ary operation. We call an $n \times m$ matrix $X = (x_{i,j})$ over A a **ϱ -matrix** if the rows of X belong to ϱ , i.e., $(x_{i,1}, \dots, x_{i,m}) \in \varrho$ for all i ($1 \leq i \leq n$). The operation f is said to **preserve** the relation ϱ if for every ϱ -matrix $(x_{i,j}) \in A^{n \times m}$ the m -tuple

$$(f(x_{1,1}, \dots, x_{n,1}), \dots, f(x_{1,m}, \dots, x_{n,m}))$$

also belongs to ϱ . It is obvious that the operation f preserves the relation ϱ if and only if ϱ is a subalgebra of the algebra $(A; f)^m$.

For a subset R of \mathcal{R}_A the set of all finitary operations on A that preserve each member of R will be denoted by $\text{Pol}(R)$. If R is finite, say $R = \{\varrho_1, \dots, \varrho_s\}$, then we simply write $\text{Pol}(\varrho_1, \dots, \varrho_s)$. On the other hand, for a subset F of \mathcal{O}_A the set of all finitary relations on A that are preserved by each member of F will be denoted by $\text{Inv}(F)$. If F is finite, say $F = \{f_1, \dots, f_s\}$, then we simply write $\text{Inv}(f_1, \dots, f_s)$.

For every finite set A the maps

$$\begin{aligned} \text{Inv}: P(\mathcal{O}_A) &\rightarrow P(\mathcal{R}_A), & F &\mapsto \text{Inv}(F), \\ \text{Pol}: P(\mathcal{R}_A) &\rightarrow P(\mathcal{O}_A), & R &\mapsto \text{Pol}(R) \end{aligned}$$

define a Galois connection between sets of operations and sets of relations.

Monoidal intervals

We will always assume that the base set A is a finite set with more than one element. To give a more detailed introduction into the concept of a monoidal interval let M be a transformation monoid on A , and let $\text{Int}(M)$ denote the collection of all clones \mathcal{C} on A such that the set of unary operations of \mathcal{C} is M . The clone $\langle M \rangle$ of essentially unary operations generated by M is a member of $\text{Int}(M)$, in fact, it is the least member of $\text{Int}(M)$, so $\text{Int}(M)$ is non-empty. Furthermore, it is clear that every clone \mathcal{C} in $\text{Int}(M)$ is contained in the set

$$\text{Sta}(M) = \left\{ f(x_1, \dots, x_n) \in \mathcal{O}_A \mid n \in \mathbb{N}, \text{ and} \right. \\ \left. f(m_1(x), \dots, m_n(x)) \in M \text{ for all } m_1, \dots, m_n \in M \right\},$$

which is called the **stabilizer** of the monoid M . It is easy to verify that $\text{Sta}(M)$ is a clone on A , therefore $\text{Sta}(M)$ is the largest member of $\text{Int}(M)$. Moreover, we see that a clone $\mathcal{C} \in \text{CL}_A$ belongs to $\text{Int}(M)$ if and only if $\langle M \rangle \subseteq \mathcal{C} \subseteq \text{Sta}(M)$. Thus $\text{Int}(M)$ is the interval $[\langle M \rangle, \text{Sta}(M)]$ in the clone lattice CL_A . Such an interval is called a **monoidal interval**.

With the help of Post's theorem one can easily describe the monoidal intervals in CL_A for $A = \{0, 1\}$. There are six monoidal intervals, and exactly the monoidal intervals corresponding to the monoids $\{\text{id}_A, c_0, c_1, r\}$, $\{\text{id}_A, c_0, c_1\}$, and $\{\text{id}_A, r\}$, respectively, are finite. The remaining three monoidal intervals, which correspond to the monoids $\{\text{id}_A\}$, $\{\text{id}_A, c_1\}$, and $\{\text{id}_A, c_0\}$, have cardinality \aleph_0 .

Recall from the introduction that if a monoidal interval $\text{Int}(M)$ has only one element, then the transformation monoid M is called **collapsing**. In this case the only element of $\text{Int}(M)$ is $\langle M \rangle$. By a result of J.-U. Grabowski [Gra97], M is collapsing if and only if $\text{Sta}(M)$ contains no essentially binary operations. Hence it is decidable for a monoid M whether M is collapsing. However, since there are at least $|A|^{|A|^2} - 2 \cdot |A|^{|A|} > 0.99 \cdot |A|^{|A|^2}$ essentially binary operations on A if $|A| \geq 3$, therefore in practice it is rather difficult to decide whether or not a monoid is collapsing.

There is a simple but useful necessary condition for a transformation monoid M on A to be collapsing: if M is collapsing, then M is **weakly transitive**, that is, there is an element $a \in A$ such that $\{m(a) : m \in M\}$ coincides with A (T. Ihringer and R. Pöschel [IP93]).

For further reference we state Pálffy's theorem that was mentioned in the introduction.

Theorem 1 (cf. P. P. Pálffy [Pal84]). *Let A be a finite set with $|A| \geq 3$, and let M be a transformation monoid on A that contains all the unary constant*

operations and whose nonconstant operations are permutations. Then $|\text{Int}(M)| \leq 2$; moreover, $|\text{Int}(M)| = 1$ unless M coincides with the monoid of all unary polynomial operations of a finite vector space over a finite field.

To prove that for a transformation monoid M the monoidal interval $\text{Int}(M)$ has cardinality 2^{\aleph_0} the method of J. Demetrovics and L. Hannák in [DH97] will be useful.

3. LARGE INTERVALS OF COLLAPSING MONOIDS

As the title indicates, we will prove that in the submonoid lattice of the full transformation semigroup on a finite set with at least 6 elements there are ‘large’ intervals such that all of their members are collapsing. Now, we describe the construction that leads to these monoids, which works for an on at least four-element set.

Let A be a finite set with at least 4 elements. Let P, Q , and R be pairwise disjoint nonempty subsets of A such that $|R| \geq 2$. Let $T(P, Q, R)$ be the set of all transformations $t \in T(A)$, such that for all $p \in P, q \in Q$ and $r, r' \in R$ if $t(r) = t(r')$ then $t(p) \in \{t(q), t(r)\}$. Let M be an arbitrary transformation monoid on A . The monoid M is said to be **rich** with respect to P, Q, R if for some $s \in A$, and for all $a, b \in A$ such that $a \neq b$ and $s \in \{a, b\}$, M contains transformations m and n such that $m(P) = m(Q) = \{a\}$, $m(R) = \{b\}$ and $n(P) = n(R) = \{a\}$, $n(Q) = \{b\}$.

The following theorem shows the importance of rich monoids.

Theorem 2 ([Dor02]). *Let A be a finite set with at least four elements, and let P, Q, R be disjoint nonempty subsets of A such that $|R| \geq 2$. Then every rich monoid $M \subseteq T(P, Q, R)$ is collapsing.*

This theorem allows us to construct ‘large’ intervals consisting of collapsing monoids.

Let A be a finite set with $|A| \geq 6$. Let the elements $p, q, r, r' \in A$ be pairwise distinct, and let $P = \{p\}$, $Q = \{q\}$, $R = \{r, r'\}$, $A' = A \setminus (P \cup Q \cup R)$. We define the monoid N on A to be the monoid generated by the set of all transformations $t \in T(P, Q, R)$ for which $t(r) = t(r')$ and the restriction of t onto A' is the identity operation on A' . It is easy to see that N is contained in $T(P, Q, R)$. For an arbitrary monoid $K \in T(A')$ we will denote by \hat{K} the monoid which consists of all transformations from $T(A)$ whose restriction onto A' is a member of K , and whose restriction onto the set $P \cup Q \cup R$ is the identity operation. Since

$t \in \langle N \cup \widehat{K} \rangle$ implies that $t|_{A'} \in K$, we get that if K_1, K_2 are submonoids of $T(A')$ and $K_1 \neq K_2$ then $\langle N \cup \widehat{K}_1 \rangle \neq \langle N \cup \widehat{K}_2 \rangle$. Furthermore, $\langle N \cup \widehat{T(A')} \rangle \subseteq T(P, Q, R)$, and N is rich.

Using the fact that the cardinality of the submonoid lattice of the full transformation semigroup on A is greater than $2^{2^{c|A|}}$ for some positive constant c and Theorem 2 we get the following theorem.

Theorem 3 ([Dor02]). *Let A be a finite set with $|A| = n \geq 6$. Then all members of the interval $[N, \langle N \cup \widehat{T(A')} \rangle]$ is collapsing, and this interval has cardinality greater than $2^{2^{c'n}}$ for some positive constant c' .*

On a 3-element set we can not use the previous construction, however, a similar method will work in this case.

Let A be a 3-element set. We will define two sets of transformations on A . Let $p, s \in A$ be arbitrary elements of A . Let T_p denote the set of all transformations $t \in T(A)$ such that either t is a permutation fixing p or t is not a permutation, and $t(p) \in \{t(q), t(r)\}$, where $\{p, q, r\} = A$. Furthermore, let $M_{p,s}$ be the set of all transformations $t \in T_p$ such that $t(A) \subseteq \{s, a\}$ for some $a \in A \setminus \{s\}$ or t is the identity operation. It is easy to see that both T_p and $M_{p,s}$ are transformation monoids on A .

With the aid of these monoids we get a similar description as in Theorem 2.

Theorem 4 ([Dor02]). *Let A be a 3-element set. Then each monoid M for which there are elements $p, s \in A$ such that $M_{p,s} \subseteq M \subseteq T_p$ is collapsing.*

Let \bowtie be the relation on $T(A)$ defined in the following way: the transformation monoids M_1 and M_2 on A are \bowtie -related if there is permutation π for which $M_2 = \{\pi^{-1}m\pi : m \in M_1\}$ hold. It is easy to see that \bowtie is an equivalence relation, and the monoidal intervals that correspond to \bowtie -related monoids are isomorphic. On a 3-element set there are 699 monoids in 160 \bowtie -classes. The last theorem in this section characterizes the collapsing monoids among them.

Theorem 5 ([Dor02]). *On the 3-element set $A = \{0, 1, 2\}$ there are 30 collapsing monoids in 11 \bowtie -classes. If M is a collapsing monoid on A , then M is equivalent to exactly one of the following monoids:*

- (1) $\langle c_0, \tau_2 \rangle = \{\text{id}_A, c_0, c_1, \tau_2\}$,
- (2) $\langle c_0, c_1, c_2 \rangle = \{\text{id}_A, c_0, c_1, c_2\}$,
- (3) $\langle c_0, c_2, \tau_2 \rangle = \{\text{id}_A, c_0, c_1, c_2, \tau_0\}$,

- (4) $\langle c_0, \sigma \rangle = \{\text{id}_A, c_0, c_1, c_2, \sigma, \sigma^2\}$,
- (5) S_3 ,
- (6) $M_{2,0}$,
- (7) $M_{2,2}$,
- (8) $\langle M_{2,2} \cup \{\tau_2\} \rangle = M_{2,2} \cup \{\tau_2\}$,
- (9) $T_2 \setminus \{\tau_2\}$,
- (10) T_2 ,

where T_2 is the monoid of all transformations $t \in T(A)$ such that either $t = \text{id}_A$ or $t(2) \in \{t(0), t(1)\}$, while $M_{2,r}$ ($r \in \{0, 2\}$) is the monoid of all transformations $t \in T_0$ for which either $|t(A)| \leq 2$ and $r \in t(A)$ or $t = \text{id}_A$ or t is constant.

4. COLLAPSING INVERSE MONOIDS

For permutation groups the results known so far indicate that ‘large’ permutation groups, e.g. all primitive permutation groups, are collapsing (cf. Pálffy–Szendrei [PSz82] and Kearnes–Szendrei [KSz01]). This motivated us in extending the investigation of collapsing monoids to ‘large’ inverse monoids.

To formulate our results we need some definitions and concepts.

Let $\mathbb{L} = (L; \vee, \wedge)$ be a finite lattice. The least and greatest elements of \mathbb{L} will be denoted by 0 and 1, respectively. The set of atoms and the set of join-irreducible elements of \mathbb{L} will be denoted by $\mathcal{A}(\mathbb{L})$ and $\mathcal{J}(\mathbb{L})$, respectively, and we put $\mathcal{A}_0(\mathbb{L}) = \mathcal{A}(\mathbb{L}) \cup \{0\}$. If there is no danger of confusion, we simply write \mathcal{A} , \mathcal{A}_0 and \mathcal{J} , respectively. Two elements a and b of \mathbb{L} will be called **similar** if and only if the principal ideals $(a]$ and $(b]$ are isomorphic. We write $a \sim b$ to denote that a is similar to b . The relation \sim is an equivalence relation on L . If the \sim -class containing a has only one element then a will be called **isolated**. For every element $a \in L$ we define a unary operation φ_a by the rule $\varphi_a(x) = x \wedge a$ ($x \in L$). In particular, φ_0 is constant with range $\{0\}$. For similar elements $a, b \in L$ the symbol $\beta_{a,b}$ will denote an isomorphism between the principal ideals $(a]$ and $(b]$. Define a set $\text{IS}(\mathbb{L})$ of transformations on L in the following way:

$$\text{IS}(\mathbb{L}) = \{\beta_{v,w} \circ \varphi_v \mid v, w \in L, v \sim w, \text{ and } \beta_{v,w}: (v] \rightarrow (w] \text{ is an isomorphism}\}.$$

Then $\text{IS}(\mathbb{L})$ is an inverse submonoid of the full transformation semigroup on L (cf. Saito–Katsura [SK92], Lemma 3.1).

First, we need the following definition. Let a and b be arbitrary elements of L . We will say that the element b is **dwarfed** by a if for all elements $b' \in L$ such that $b' \sim b$ we have that $b' \leq a$. We will use the notation $b \ll a$ to denote that a dwarfs b . Now we are in a position to state the central result of this section.

Theorem 6 ([Dor07]). *Let \mathbb{L} be a finite lattice such that $|L| \geq 3$. Then the inverse monoid $M = \text{IS}(\mathbb{L})$ is collapsing if and only if no element of $\mathcal{J} \setminus \mathcal{A}$ dwarfs a nonzero element of L .*

If \mathbb{L} is an atomistic lattice then $\mathcal{J} = \mathcal{A}$, and so, the conditions of the previous theorem are satisfied.

Corollary 7. *If \mathbb{L} is an atomistic lattice then $\text{IS}(\mathbb{L})$ is collapsing.*

Describing lattices $\mathbb{L}L$ for which $\text{IS}(\mathbb{L})$ is collapsing we turn our attention to large monoidal intervals. We conclude this section with a discussion of lattices \mathbb{L} for which the monoidal interval $\text{Int}(\text{IS}(\mathbb{L}))$ has cardinality 2^{\aleph_0} . For elements $u \leq v$ of \mathbb{L} , we will use the notation $[u, v]$ for the interval $\{x \in L \mid u \leq x \leq v\}$. We will call a lattice \mathbb{L} **pinched** if L contains an element $b \in L \setminus \{0, 1\}$ such that $L = [0, b] \cup [b, 1]$.

Next theorem services as a basis for further constructions.

Theorem 8 ([Dor07]). *Let \mathbb{L} be a pinched lattice, and let $b \in L \setminus \{0, 1\}$ be an element such that $L = [0, b] \cup [b, 1]$. Then $|\text{Int}(\text{IS}([0, b]))| \leq |\text{Int}(\text{IS}(\mathbb{L}))|$.*

The most natural examples for pinched lattices are finite chains with at least 3 elements.

Theorem 9 ([Dor07]). *For a 3-element chain \mathbb{L} we have $|\text{Int}(\text{IS}(\mathbb{L}))| = 2^{\aleph_0}$.*

Corollary 10. *If \mathbb{L} is a finite chain with at least 3 elements then the monoidal interval $\text{Int}(\text{IS}(\mathbb{L}))$ has cardinality 2^{\aleph_0} .*

Finally, combining Post's results for description of $\mathbb{C}\mathbb{L}_{\{0,1\}}$ and Theorem 8, we can state the following.

Corollary 11. *If \mathbb{L} is a finite lattice which has a unique atom then the monoidal interval $\text{Int}(\text{IS}(\mathbb{L}))$ is infinite.*

5. COLLAPSING MONOIDS CONSISTING OF CONSTANTS AND PERMUTATIONS

The well-known result of P. P. Pálffy, Theorem 1, inspired the investigation of monoids consisting of constants and permutations.

We restrict our efforts to monoids that contain at least one but not all unary constant operations and whose nonconstant operations are permutations. We need further definitions to form our results.

Let V be the set of all elements $v \in A$ such that $c_v \in M$, and set $W = A \setminus V$. Assume that $\emptyset \subsetneq V, W \subsetneq A$. Define P to be the set of all permutations contained in M . The facts that A is finite and M is closed under composition ensure that P is a permutation group on A and

$$\alpha(V) = V, \quad \alpha(W) = W$$

hold for all $\alpha \in P$. These equalities allow us to restrict P to V and W , and obtain the permutation groups

$$P_V = \{\alpha|_V \mid \alpha \in P\} \subseteq S(V),$$

$$P_W = \{\alpha|_W \mid \alpha \in P\} \subseteq S(W).$$

Furthermore, let i_V be the restriction map $i_V: P \rightarrow P|_V, \alpha \mapsto \alpha|_V$. If the map i_V is injective, then for every transformation $m \in M$ the unique extension of the map $m|_V$ to A is m . Hence, if the map i_V is injective, the map

$$j: P_V \rightarrow P_W, \alpha|_V \mapsto \alpha|_W.$$

is well-defined.

Our first theorem characterizes all collapsing monoids that consist of permutations and at least one unary constant operation. This extends the results obtained by A. Fearnley and I. Rosenberg in [FR03].

Theorem 12 ([Dor08]). *Let A be a finite set with at least two elements, and let M be a transformation monoid on A that consists of at least one unary constant operation and some permutations. Then M is collapsing if and only if*

- (i) $|V| \geq 2$,
- (ii) P_W is transitive,
- (iii) i_V is injective, and

(iv) one of the following conditions holds:

- (a) the monoid $M|_V$ is collapsing,
- (b) the map j is not injective,
- (c) the permutation group P_W is not regular.

As a consequence of the previous theorem we get that a monoid $M \subseteq C(A) \cup S(A)$ containing exactly one unary constant operation cannot be collapsing. The following theorem states a bit more.

Theorem 13 ([Dor08]). *Let $M \subseteq C(A) \cup S(A)$ be a monoid such that it contains only one unary constant operation. Then the monoidal interval $\text{Int}(M)$ is infinite.*

Theorem 14 ([Dor08]). *Let A be a finite set with at least two elements, and let M be a transformation monoid on A that consists of at least two unary constant operations and some permutations. If conditions (i)–(iii) of Theorem 12 hold but condition (iv) of Theorem 12 fails for M then $\text{Int}(M)$ is isomorphic to $\text{Int}(M|_V)$. Hence,*

- if $|V| = 2$ and $M|_V$ is the monoid $\{\text{id}_V, c_0|_V, c_1|_V\}$, then $|W| = 1$, $M = \{\text{id}_A, c_0, c_1\}$, and $\text{Int}(M)$ is isomorphic to the direct square of the 2-element chain;
- if $|V| = 2$ and $M|_V$ is the full transformation semigroup on V , then $|W| = 2$, $M = \{\text{id}_A, c_0, c_1, (01)(23)\}$, and $\text{Int}(M)$ is a 3-element chain;
- if $|V| \geq 3$, then $\text{Int}(M)$ is a 2-element chain.

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