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Binary Tomography Using Geometrical Priors: Uniqueness and Reconstruction Results

Summary of the PhD Dissertation

by

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Introduction

Computerized tomography (CT) is a diagnostic procedure for obtaining the density distribution within the human body from its x-ray projections, which makes it possible to recognize disorders within certain human organs. In the last few decades applications of CT have spread to a wide area beyond medical imaging. These include fields of industry, biology, physics and chemistry. In most of these new applications the number of the possible density values of the object in investigation is small, but there is a practical limitation that only a few projections of the object can be made. In contrast, in medical imaging the density values can vary across a wide range and the number of available projections is usually a few hundred. Due to the very limited number of projections in the new applications the usual classical reconstruction methods of CT do not work that well.

Discrete tomography (DT) [14; 15] investigates how, with prior knowledge, we can reconstruct an object. The reconstruction should only contain a few values that can be exploited to eliminate the problems arising from using a small number of available projections. A more restrictive but still very common case is when the density values can only be elements of the set $\{0, 1\}$. For example, in electron microscopy 0 and 1 can represent the absence and presence of a certain atom in the crystalline structure, respectively. Similarly, in angiography the values 0 and 1 can describe the absence or presence of a contrast agent in heart chambers or in segments of blood vessels.

This dissertation is concerned with this latter field called *binary tomography* (BT). The main challenge in BT is that practical limitations every time reduce the number of available projections to at most about ten (usually far fewer) – which results in ambiguous reconstruction, i.e. the number of possible solutions of the same reconstruction task can be extremely large. This can cause the reconstructed discrete set to be quite unlike to the original one. In addition, the reconstruction problem can be NP-hard, depending on the number and directions of the projections. One way of eliminating these problems is to use metaheuristics (like simulated annealing and genetic algorithms) to find possibly good but not necessarily exact solutions. Another strategy is to suppose that the set to be reconstructed has some geometrical properties. In this way we can reduce the search space of the possible solutions and we can achieve fast and rare ambiguous reconstructions. This thesis summarizes the author's research results in reconstructing discrete sets that have certain geometrical properties.

Binary Tomography: Definitions and Basic Problems

The finite subsets of the 2D integer lattice $\mathbb{Z}^2$ are called *discrete sets*. The *size* of a discrete set is the size of its minimal bounding discrete rectangle. A discrete set F of size $m \times n$ is defined up to a translation and it can be represented by a binary picture formed from unitary cells or by a binary matrix $\hat{F} = (\hat{f}_{ij})_{m \times n}$. To be consistent with the corresponding matrix representation we shall assume that the vertical axis of the 2D integer lattice is top-down directed and that the upper left corner of the minimal bounding rectangle of F is the $(1, 1)$ position. For the same reason we will refer to any element of a discrete set by its matrix position (i.e. not by the position of the element in the 2D integer lattice).

A *lattice direction* v is represented by a non-zero vector $(a, b) \in \mathbb{Z}^2$ such that a and b are coprimes. A *lattice line* in direction v is a line in the 2D Euclidean space $\mathbb{E}^2$ that is parallel to v and passes through at least one point of $\mathbb{Z}^2$. Now let us denote the set of lattice lines in direction v by $\mathcal{L}^{(v)}$. Then the *projection* of the discrete set F in direction v is defined as the function $\mathcal{P}_F^{(v)} : \mathcal{L}^{(v)} \to \mathbb{N}_0$ where $\mathcal{P}_F^{(v)}(\ell) = |F \cap \ell|$ for each $\ell \in \mathcal{L}^{(v)}$. Projections of a discrete set outside its minimal bounding discrete rectangle always have a value of 0 so they are not interesting in our study.

Given an arbitrary class $\mathcal{G}$ of discrete sets we say that the discrete set $G \in \mathcal{G}$ is *uniquely determined* in the class $\mathcal{G}$ (with respect to some projections) if there is no other discrete set $G' \in \mathcal{G}$ with the same projections. Now three main problems arise in discrete tomography. Given an arbitrary class of discrete sets $\mathcal{G}$ and an arbitrary set $\mathcal{L} = (v_1, \dots, v_q)$ of q lattice directions these problems can be stated as follows.

Consistency($\mathcal{G}$, $\mathcal{L}$)

Instance: *For $k = 1, \dots, q$ a function $p^{(k)} : \mathcal{L}^{(v_k)} \to \mathbb{N}_0$ with finite support.*

Task: *Decide whether there exists a discrete set $F \in \mathcal{G}$ such that $\mathcal{P}_F^{(v_k)} = p^{(k)}$ for $k = 1, \dots, q$.*

Reconstruction($\mathcal{G}$, $\mathcal{L}$)

Instance: *For $k = 1, \dots, q$ a function $p^{(k)} : \mathcal{L}^{(v_k)} \to \mathbb{N}_0$ with finite support.*

Task: *Construct a discrete set $F \in \mathcal{G}$ such that $\mathcal{P}_F^{(v_k)} = p^{(k)}$ for $k = 1, \dots, q$.*

Uniqueness($\mathcal{G}$, $\mathcal{L}$)

Instance: *An $F \in \mathcal{G}$.*

Task: *Decide whether F is uniquely determined in the class $\mathcal{G}$ with respect to its projections in the directions of $\mathcal{L}$.*

In this thesis we use two special sets of directions, namely $\mathcal{L}_2 = \{(1,0),(0,1)\}$ and $\mathcal{L}_4 = \{(1,0),(0,1),(1,1),(-1,1)\}$. We introduce the notations $\mathcal{H}(F) = (h_1, \dots, h_m)$, $\mathcal{V}(F) = (v_1, \dots, v_n)$, $\mathcal{D}(F) = (d_1, \dots, d_{m+n+1})$, and $\mathcal{A}(F) = (a_1, \dots, a_{m+n-1})$ for the projections of a discrete set F in the $(1,0)$, $(0,1)$, $(1,1)$, and $(-1,1)$ directions, respectively. We will refer to them as the *horizontal, vertical, diagonal*, and *antidiagonal projections* of F, respectively. We also will use the *cumulated vectors* of F. They are represented by the vectors $\widetilde{H} = (\widetilde{h}_1, \dots, \widetilde{h}_m)$, $\widetilde{V} = (\widetilde{v}_1, \dots, \widetilde{v}_n)$, $\widetilde{D} = (\widetilde{d}_1, \dots, \widetilde{d}_{m+n-1})$, and $\widetilde{A} = (\widetilde{a}_1, \dots, \widetilde{a}_{m+n-1})$, and defined by $\widetilde{h}_i = \sum_{l=1}^{i} h_l$, $\widetilde{v}_j = \sum_{l=1}^{j} v_l$, $\widetilde{d}_k = \sum_{l=1}^{k} d_l$, and $\widetilde{a}_k = \sum_{l=1}^{k} a_l$.

In the thesis we examine the Reconstruction($\mathcal{G}$, $\mathcal{L}$) and Uniqueness($\mathcal{G}$, $\mathcal{L}$) tasks for several classes of discrete sets using the direction sets $\mathcal{L}_2$ and $\mathcal{L}_4$. To facilitate the reconstruction task we always assume that the discrete set to be reconstructed has some geometrical properties, like connectedness, convexity and directedness.

Two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in a discrete set F are said to be *4-adjacent* if $|p_1 - q_1| + |p_2 - q_2| = 1$. The points P and Q are said to be *8-adjacent* if they are 4-adjacent or $(|p_1 - q_1| = 1$ and $|p_2 - q_2| = 1)$. The sequence of distinct points $P_0, \dots, P_k$ is a *4/8-path* from point P_0 to point P_k in a discrete set F if each point of the sequence is in F and P_l and P_{l-1} are 4/8-adjacent, respectively, for each $l = 1, \dots, k$. A discrete set F is *4/8-connected* if, for any two points of F, there is a 4/8-path in F, respectively, between them. The 4-connected set is also known as a *polyomino*. If the discrete set is not 4-connected then it consists of several polyominoes. The maximal 4-connected subsets of a discrete set F give a uniquely determined partition of F. Such a maximal 4-connected subset of a discrete set F is called a *component* of F.

The discrete set F is *convex along the direction* $v = (a, b)$ if for any two points $(i_1, j_1) \in F$ and $(i_2, j_2) \in F$ if $(i_2, j_2) = (i_1, j_1) + k \cdot v$ for some $k \in \mathbb{Z}$ then $(i_1, j_1) + t \cdot v \in F$ for every $t \in \{0, \ldots, k\}$ as well. In particular, the discrete set is horizontally/vertically/diagonally/antidiagonally convex if it is convex along the $(1, 0)/(0, 1)/(1, 1)/(-1, 1)$ direction, respectively. If a discrete set is both horizontally and vertically convex it is then called hv-convex. We introduce the notations $\mathcal{HV}$ and $\mathcal{S}'_8$ for the class of hv-convex and hv-convex 8- but not 4-connected discrete sets, respectively.

For a point $P = (p_1, p_2)$ we can define the four *quadrants around* P by

$$\begin{aligned}
R_0(P) &= \{Q = (q_1, q_2) \mid q_1 \leq p_1 \text{ and } q_2 \leq p_2\} , \\
R_1(P) &= \{Q = (q_1, q_2) \mid q_1 \geq p_1 \text{ and } q_2 \leq p_2\} , \\
R_2(P) &= \{Q = (q_1, q_2) \mid q_1 \geq p_1 \text{ and } q_2 \geq p_2\} , \\
R_3(P) &= \{Q = (q_1, q_2) \mid q_1 \leq p_1 \text{ and } q_2 \geq p_2\} .
\end{aligned} \tag{1}$$

A discrete set F is *Q-convex* if $R_k(P) \cap F \neq \emptyset$ for all $k \in \{0, 1, 2, 3\}$ implies $P \in F$. We will denote the class of Q-convex sets which have several components by $\mathcal{Q}'$.

A 4-path in a discrete set F is a *northeast path* (NE-path for short) from point P_0 to point P_t if each point P_l of the path is north or east of P_{l-1} for each $l = 1, \ldots, t$. SW-, SE-, NW-paths can be similarly defined. The discrete set F is *NE-directed* if there is a particular point of F, called the *source*, such that there is an NE-path in F from the source to any other point of F. Similar definitions can be given for SW-, SE-, and NW-directedness. We also say that the discrete set is *directed* if it is NE-, SW-, SE-, or NW-directed. For a given direction (a, b) the class of NE-directed polyominoes which are convex along the (a, b) direction will be denoted by $\mathcal{DCP}^{NE}_{(a,b)}$.

(Non)-Uniqueness Results for Directed Polyominoes

Consider the task of reconstructing a polyomino such that its horizontal and vertical projections are equal to a given vector $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$, respectively. As was shown in [12], the assumption that the polyomino in question is convex along at least one of the horizontal and vertical directions, and that it is also NE-directed can ensure that the solution of the above problem is uniquely determined. Later an algorithm was also presented for reconstructing polyominoes like this of size $m \times n$ using their horizontal and vertical projections in $O(mn)$ time [17]. From the constructions given in [12] and [17] it also follows that SE-, NW- or SW-directed polyominoes which are horizontally or vertically convex can also be reconstructed uniquely and with the same time complexity. These results on the uniqueness and reconstruction of directed polyominoes can be summarized in the following

Theorem 1 *[12; 17] Every horizontally or vertically convex directed polyomino of size $m \times n$ can be reconstructed from its source and its horizontal and vertical projections uniquely in $O(mn)$ time.*

Now, we will examine whether Theorem 1 can be generalized to other directions of convexity. We will see how the change of direction of convexity affects the number of possible solutions of the same reconstruction task. Although we will only present results for NE-directed polyominoes here, there is a straightforward way of getting similar results for SE-, NW or SW-directed polyominoes as well.

We first will focus on diagonally convex NE-directed polyominoes, i.e. on the class $\mathcal{DCP}^{NE}_{(1,1)}$. The following simple lemma holds for an arbitrary NE-directed polyomino.

Lemma 1 *Let D be a NE-directed polyomino with $\mathcal{H}(D) = (h_1, \ldots, h_m)$ and $\mathcal{V}(D) = (v_1, \ldots, v_n)$. Then $(m, j) \in D$ if and only if $1 \leq j \leq h_m$, and $(i, 1) \in D$ if and only if $m - v_1 < i \leq m$.*

Now consider an arbitrary polyomino $D \in \mathcal{DCP}^{NE}_{(1,1)}$. From Lemma 1, the subset F of the polyomino D (which consists of all the elements of the first column and the last row of D) is determined by the vector components h_m and v_1. On the basis of the following lemma the remaining elements of D can be determined by the set F.

Lemma 2 *Let $D \in \mathcal{DCP}^{NE}_{(1,1)}$, $F \subset D$ and $(i, j) \in \{1, \ldots, m-1\} \times \{2, \ldots, n\}$ be a position such that for every $(i', j') \neq (i, j)$ if $i' \geq i$ and $j' \leq j$ then $(i', j') \in D \leftrightarrow (i', j') \in F$. Then $\sum_{t=i+1}^{n} \hat{f}_{tj} < v_j$ and $\sum_{t=1}^{j-1} \hat{f}_{it} < h_i$ are necessary and sufficient conditions for $(i, j) \in D$.*

That is when $D \in \mathcal{DCP}^{NE}_{(1,1)}$ then, from Lemma 1, the first column and the last row of D are uniquely determined by v_1 and h_m, respectively, i.e. a subset F of the polyomino D can be found (consisting of all the positions of the last row and first column of D). Then for the position $(m-1, 2)$ the conditions of Lemma 2 hold. Therefore, using Lemma 2 we can establish whether the position $(m-1, 2)$ belongs to D and, if so, we set $F = F \cup \{(m-1, 2)\}$. Taking each position bottom up and left to right, F always satisfies the conditions of Lemma 2 and so the above procedure can be repeated. If H and V are the projections of a diagonally convex NE-directed polyomino then we will eventually get $F = D$. This construction also guarantees uniqueness. Thus, we can say the following

Theorem 2 *Let $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$. In the class $\mathcal{DCP}^{NE}_{(1,1)}$ there is at most one polyomino D such that $\mathcal{H}(D) = H$ and $\mathcal{V}(D) = V$.*

In addition, an algorithm (called Algorithm DCP) can also be described to reconstruct the possibly existing polyomino of the class $\mathcal{DCP}^{NE}_{(1,1)}$ with given horizontal and vertical projections in $O(mn)$ time.

However, in contrast to Theorem 2, we find that there is a drastic change in the number of NE-directed polyominoes which have the same horizontal and vertical projections if, instead of diagonal convexity, it is assumed that the polyomino is antidiagonally convex.

Theorem 3 *In the class $\mathcal{DCP}^{NE}_{(-1,1)}$ there can be exponentially many polyominoes with the same horizontal and vertical projections.*

From Theorem 2 and Theorem 3 it is quite apparent that the direction of convexity plays an important role in determining whether ambiguity can be eliminated. We also find that Theorem 3 can be generalized to polyominoes which are convex along an arbitrary lattice direction $d \notin \{(1, 0), (0, 1), (1, 1)\}$.

Theorem 4 *Let $d = (a, b)$ be a lattice direction such that $d \notin \{(1, 0), (0, 1), (1, 1)\}$. In the class $\mathcal{DCP}^{NE}_{(a,b)}$ there can be exponentially many polyominoes with the same horizontal and vertical projections.*

We should also add, that using Theorem 2 and Theorem 4 we can generalise our results to NE-directed polyominoes which satisfy convexity properties along an arbitrary set of finite directions, and also for some infinite sets of directions as well.

Our theoretical results concerning uniqueness and non-uniqueness of directed polyominoes and the reconstruction algorithm were published in [1].

Reconstruction of Q-Convex non-4-Connected Discrete Sets

The class of the Q-convexes is one of the broadest known classes in which polynomial-time reconstruction is possible using just two projections. As was shown in [11], the reconstruction of discrete sets that are Q-convex along the horizontal and vertical directions can be performed in $O(N^4 \log N)$ time (where $N = \max\{m, n\}$) using horizontal and vertical projections. In our thesis we present a faster algorithm for this class when it is also known in advance that the Q-convex set to be reconstructed consists of two or more components. An important observation here is that for an arbitrary set of $\mathcal{Q}'$ the smallest containing discrete rectangles of the components can be arranged in just two possible ways. Omitting possible empty rows and columns they are connected to each other by their bottom right and upper left, or by their bottom left and upper right positions. In the former case we say that the set is of *NW type*, in the latter case we say that the set is of *NE type*. The following lemma is about the directedness of the components, which depends on the type of the set.

Lemma 3 *Let $F \in \mathcal{Q}'$ that has components $F_1, \ldots, F_k$ $(k \geq 2)$. If F is of NW/NE type then $F_1, \ldots, F_{k-1}$ are NW/NE-directed and $F_2, \ldots, F_k$ are SE/SW-directed, respectively.*

Now we will demonstrate how we can represent a set of $\mathcal{Q}'$. The following corollary can be stated.

Corollary 1 *Let $F \in \mathcal{Q}'$ which has components $F_1, \ldots, F_k$. Then there are uniquely determined row indices $0 < i_1 < \cdots < i_k = m$ and column indices $0 < j_1 < \cdots < j_k \leq n$ such that for each $l = 1, \ldots, k$ $(k \geq 2)$ (i_l, j_l) is the bottom-right position of the smallest containing discrete rectangle (SCDR) of F_l when F is of NW type and (i_l, j_{k-l+1}) is the bottom-left position of the SCDR of F_l when F is of NE type.*

Depending on the type of F, let us define

$$C_F = \begin{cases} \{(i_l, j_l) \mid l = 1, \ldots, k-1\}, & \text{when } F \text{ is of NW type}\ , \\ \{(i_l, j_{k-l+1}) \mid l = 1, \ldots, k-1\}, & \text{when } F \text{ is of NE type}\ , \end{cases} \tag{2}$$

where $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ denote the uniquely determined indices mentioned in Corollary 1. It is not hard to see that C_F consists of the source points of the NW-/NE-directed components $F_1, \ldots, F_{k-1}$ when F is of NW/NE type, respectively. The knowledge of any element of C_F is useful in the reconstruction of a set $F \in \mathcal{Q}'$, which can be stated in following theorem.

Theorem 5 *Any $F \in \mathcal{Q}'$ is uniquely determined in $\mathcal{Q}'$ by its horizontal and vertical projections, its type, and an arbitrary element of C_F.*

Using Theorem 5 we can reconstruct the Q-convex set F from its horizontal and vertical projections if we know the type of F and at least one element of C_F. We find that the cumulated vectors of F are appropriate for determining elements of C_F. We say that the position $(i, j) \in \{1, \ldots, m-1\} \times \{1, \ldots, n-1\}$ is an *equality position of NW type* of F if $\widetilde{h}_i = \widetilde{v}_j$. In an analogous way, we say that $(i, j) \in \{1, \ldots, m\} \times \{2, \ldots, n+1\}$ is an *equality position of NE type* of F if $\widetilde{h}_i = \widetilde{v}_n - \widetilde{v}_{j-1}$. Now let us denote the set of equality positions of NW/NE type of F by L_F^{NW}/L_F^{NE}, respectively. We can prove that there is a strong connection between the set of equality positions of F and the set of source points C_F. Namely, $C_F \subseteq L_F^{NW}$ when F is of NW type and $C_F \subseteq L_F^{NE}$ when F is of NE type.

Using the above results we can describe an algorithm for reconstructing sets of $\mathcal{Q}'$ from their horizontal and vertical projections. This algorithm is called Algorithm 2-RECQ' and it verifies every equality position whether it can be the source of a component in such a way, that is simply tries to reconstruct the components from their assumed (or true) sources. As regards the analysis of the complexity of Algorithm 2REC-Q', we can state the following

Theorem 6 *Algorithm 2-RECQ' solves the* $\textsc{Reconstruction}(\mathcal{Q}', \mathcal{L}_2)$ *task in* $O(mn \cdot \min\{m, n\})$ *time. The algorithm finds all sets of* $\mathcal{Q}'$ *with the given projections.*

With some further lemmas it can also be proven that sets of $\mathcal{S}_8'$ have the same properties as sets of $\mathcal{Q}'$. The only difference is that the SCDRs of the components of a set in the class $\mathcal{Q}'$ may be separated (i.e. there may be empty rows and/or columns between two consecutive components), while in the $\mathcal{S}_8'$ class they are always 8-connected. That is,

Theorem 7 $\mathcal{S}_8' \subset \mathcal{Q}'$.

Theorem 7 immediately implies that the $\textsc{Reconstruction}(\mathcal{S}_8', \mathcal{L}_2)$ task can also be solved in $O(mn \cdot \min\{m, n\})$ time. However, there is a nice improvement in the complexity of the reconstruction when the set of $\mathcal{Q}'$ to be reconstructed is not 8-connected (i.e. if it has empty rows and/or columns). Furthermore, in this case ambiguity can only arise in the type of the discrete set. Thus, we can state

Theorem 8 *The* $\textsc{Reconstruction}(\mathcal{Q}' \backslash \mathcal{S}_8', \mathcal{L}_2)$ *task can be solved in* $O(mn)$ *time. The number of solutions is at most two.*

In [10] an algorithm (called Algorithm C) is presented which has a worst case complexity of $O(mn \cdot \min\{m^2, n^2\})$ and so far has the best average time complexity for reconstructing hv-convex 8-connected discrete sets using two projections. In order to compare the average execution times of this algorithm and Algorithm 2-RECQ' on the $\mathcal{S}_8'$ class we generated sets of $\mathcal{S}_8'$ at random from a uniform distribution. This was carried out by a slightly modified version of the method given in [10]. In our experiments we generated discrete sets of $\mathcal{S}_8'$ with different sizes. Each set of test data consisted of 1000 hv-convex 8-connected but not 4-connected discrete sets of the same size. Then we reconstructed them by using both algorithms. The average execution times in seconds for obtaining all the solutions of different test sets are presented in Table 1. The results indicate that not just the worst case complexity of our algorithm is better (see Theorem 6), but also that its average execution time is much better on all of the five test sets.

Table 1: Average execution times in seconds of Algorithm 2-REC8' and Algorithm C in [10], which both depend on the size of the discrete set

Size $n \times n$	2-REC8'	C in [10]
20×20	0.000272	0.011511
40×40	0.001064	0.032524
60×60	0.002597	0.065897
80×80	0.004746	0.116505
100×100	0.007831	0.178633

Our algorithm in its original form was described to work in the class of hv-convex 8- but not 4-connected discrete sets in [3]. The theoretical results which form the basis of the algorithm was later proved in [4]. The extended version of the algorithm which is also able to reconstruct sets of $\mathcal{Q}'$ was described in [2].

Reconstruction From Four Projections

A lot of work has been done in designing efficient reconstruction algorithms for different classes of discrete sets using just two projections. However, the theory of reconstructing discrete sets from three or more projections is, currently, far from complete. In the following we describe a method that uses four projections to reconstruct a discrete set which has at least two components. Our algorithm works for a special class of discrete sets that have disjoint components called decomposable discrete sets.

Given two discrete sets C and D represented by the binary matrices $\hat{C} = (\hat{c}_{ij})_{m_1 \times n_1}$ and $\hat{D} = (\hat{d}_{ij})_{m_2 \times n_2}$, respectively, we say that we get the discrete set F represented by the binary matrix $\hat{F} = (\hat{f}_{ij})_{m_3 \times n_3}$ by *northwest gluing* (or NW gluing in short) C to D if

$$\hat{F} = \begin{pmatrix} \hat{C} & \mathbf{0} \\ \mathbf{0} & \hat{D} \end{pmatrix} , \tag{3}$$

such that $m_3 \geq m_1 + m_2$ and $n_3 \geq n_1 + n_2$. If C is a single component then we say that C is the *NW component* of F. NE, SE, SW gluings and components are defined in a similar way.

Two components F_1 and F_2 of F are *disjoint* if both the sets of the row indices and the sets of the column indices of F_1 and F_2 are disjoint. We say that a discrete set F made up of k ($k \geq 2$) components is *decomposable* if all of the following properties are fulfilled:

(α) the components of F are uniquely reconstructible from their horizontal and vertical projections in the class of all polyominoes in polynomial time,

(β) the sets of the row and column indices of the components' smallest containing discrete rectangles (SCDR) are pairwisely disjoint,

(γ) in the case when $k > 2$ we get F by gluing a single component to a decomposable discrete set made up of $k-1$ components using one of the four gluing operators.

Actually, property (α) is usually not satisfied. But one can force the components to belong a certain class $\mathcal{C}$ of polyominoes such that every polyomino of $\mathcal{C}$ is uniquely determined in this class by its horizontal and vertical projections. In this case in the following when necessary we will say that the discrete set is decomposable w.r.t. the class $\mathcal{C}$. The type of a decomposable discrete set can be defined in much the same way as that for sets of $\mathcal{Q}'$. If by omitting empty rows and columns, the SCDRs of the components of a decomposable discrete set F are related to each other by their bottom right hand and upper left hand (bottom left hand and upper right hand) corners then we say that F is of *NW/NE type*, respectively. The class of decomposable discrete sets (w.r.t. the class $\mathcal{C}$) will be denoted by $\mathcal{DEC}$ ($\mathcal{DEC}_{\mathcal{C}}$). In addition we introduce the notations $\mathcal{S}^{NW}$, $\mathcal{S}^{NE}$ for the class of decomposable discrete sets of NW/NE type, respectively.

We can provide a description of decomposable discrete sets by stating and proving the following lemma:

Lemma 4 *A discrete set F is decomposable if and only if it satisfies property (α) and there exists a sequence of discrete sets $F^{(1)}, \ldots, F^{(k)}$ $(k \geq 2)$ such that $F^{(1)}$ consists of one component, $F^{(k)} = F$, and for each $l = 1, \ldots, k-1$ we get $F^{(l+1)}$ by gluing a component to $F^{(l)}$ using a gluing operator.*

For a given discrete set $F \in \mathcal{DEC}$ the sequence described in this lemma is not uniquely determined. But we will call any sequence which satisfies the properties of it as a *gluing sequence* of F. We also can prove that for every set $F \in \mathcal{DEC}$ there exists a unique integer j such that for every gluing sequence $F^{(1)}, \ldots, F^{(k)} = F$ $F^{(j)}$ is the same, $F^{(j)} \in \mathcal{S}^{NW} \cup \mathcal{S}^{NE}$, and $j = k$ or $F^{(j+1)} \notin \mathcal{S}^{NW} \cup \mathcal{S}^{NE}$. The uniquely determined set $F^{(j)}$ is called the *center* of F and shall be denoted by $C(F)$.

Making use of properties (α) and (β) in the reconstruction of a decomposable discrete set, it is sufficient to identify the SCDRs of the components. In the following we shall only deal with NW components. However, the results given below can be easily modified to determine NE, SE, and SW components, as well. In order to find the SCDR of a NW component we will first supply a necessary condition.

Lemma 5 *Let $F \in \mathcal{DEC}$. If (i, j) is the bottom right hand position of the SCDR of the NW-component of F then i is the smallest integer for which there exists an integer j with $\widetilde{h}_i = \widetilde{v}_j = \widetilde{a}_{i+j-1} > 0$ and $a_{i+j} = 0$.*

As luck would have it, Lemma 5 does not give a sufficient condition for identifying the SCDR of a component of an $F \in \mathcal{DEC}$. Some more conditions need to be introduced to find NW components. With the aid of the following theorem it is possible to test whether the decomposable discrete set has an NW or SE component.

Theorem 9 *Let $\mathcal{C}$ be an arbitrary class of polyominoes which can be uniquely reconstructed in this class from their horizontal and vertical projections in polynomial time. Let $F \in \mathcal{DEC}_{\mathcal{C}}$, $\mathcal{H}(F) = (h_1, \ldots, h_m)$, $\mathcal{V}(F) = (v_1, \ldots, v_n)$, and $\mathcal{A}(F) = (a_1, \ldots, a_{m+n-1})$. If (i, j) is a position that satisfies the necessary conditions of Lemma 5 such that a polyomino $P \in \mathcal{C}$ exists with $\mathcal{H}(P) = (h_1, \ldots, h_i)$, $\mathcal{V}(P) = (v_1, \ldots, v_j)$, and $\mathcal{A}(P) = (a_1, \ldots, a_{i+j-1})$ then P is the NW component of F or/and F has a SE component. If no such position exists then F has no NW component.*

If a decomposable discrete set F is in $\mathcal{S}^{NW}/\mathcal{S}^{NE}$ then with the aid of the NW/NE-version of Theorem 9 it is possible to find the SCDR of the NW/NE component of F, respectively. This means that once we have decomposed all the components around the center of F, Theorem 9 will provide an effective tool for reconstructing the center itself. On the basis of the following theorem one can find the NW component of F (if exists) if $F \in \mathcal{DEC} \setminus (\mathcal{S}^{NW} \cup \mathcal{S}^{NE})$ as well.

Theorem 10 *Assume that $F \in \mathcal{DEC} \setminus (\mathcal{S}^{NW} \cup \mathcal{S}^{NE})$ and a position (i, j) satisfies the conditions of Theorem 9 with a polyomino P. Moreover, let $\{i_1, \ldots, i_2\} \times \{j_1, \ldots, j_2\}$ be the SCDR of $C(F)$. Then P is the NW component of F if and only if there exists $i' \in \{i_1, \ldots, i_2\}$ such that $i < i'$ or there exists $j' \in \{j_1, \ldots, j_2\}$ such that $j < j'$.*

Using the necessary and sufficient condition above for finding the components, an algorithm (called Algorithm 4-RECDEC) can be developed to reconstruct decomposable discrete sets with given horizontal, vertical, diagonal, and antidiagonal projections. This algorithm reconstructs the discrete set component by component. As regards the analysis of Algorithm 4-RECDEC, we can state the following

Theorem 11 *Let $\mathcal{C}$ be an arbitrary class of polyominoes that can be reconstructed in this class uniquely from their horizontal and vertical projections in polynomial time (in $O(f(m,n))$ time, say). Then Algorithm 4-RECDEC solves the* $\textsc{Reconstruction}(\mathcal{DEC}_{\mathcal{C}},\ \mathcal{L}_4)$ *task in* $O(\min\{m,n\} \cdot f(m,n))$ *time. The algorithm finds all sets of* $\mathcal{DEC}_{\mathcal{C}}$ *with the given projections.*

We also can prove that every Q-convex set made up of at least two components is decomposable as well. In fact, a somewhat stronger relation does also hold good. Namely, $\mathcal{Q}' \subset \mathcal{S}^{NW} \cup \mathcal{S}^{NE}$. This relation has an important consequence on the issue of reconstruction complexity in the class $\mathcal{Q}'$ if we use four projections.

Corollary 2 $\textsc{Reconstruction}(\mathcal{Q}',\ \mathcal{L}_4)$ *can be solved in* $O(mn)$ *time. The solution is uniquely determined in the class* $\mathcal{Q}'$*.*

As a direct consequence of Corollary 2 and Theorem 7 we have

Corollary 3 *The* $\textsc{Reconstruction}(\mathcal{S}_8',\ \mathcal{L}_4)$ *task can be solved in* $O(mn)$ *time and its solution is uniquely determined in the class* $\mathcal{S}_8'$ *.*

Now let us study how the decomposition technique can be applied for reconstructing elements of the class of hv-convex discrete sets (denoted by $\mathcal{HV}$) using four projections. Let us first examine the class $\mathcal{HV} \cap \mathcal{DEC}$, i.e. the class of hv-convex decomposable discrete sets. The following theorem tells us that it is possible to solve the reconstruction problem in this class using four projections in polynomial time.

Theorem 12 *The algorithm 4-RECDEC solves the* $\textsc{Reconstruction}(\mathcal{HV} \cap \mathcal{DEC},\ \mathcal{L}_4)$ *task in* $O(mn \cdot \min\{m^3, n^3\})$ *time. The algorithm should find all sets of* $\mathcal{HV} \cap \mathcal{DEC}$ *with the given projections.*

That is, every hv-convex decomposable discrete set which has the same horizontal, vertical, diagonal, and antidiagonal projections can be reconstructed in polynomial time. Plainly this also means that the number of solutions is polynomial as well. The following theorem addresses the issue of whether the use of four projections is necessary to achieve this result.

Theorem 13 *For some vectors H, V, and D there can be exponentially many hv-convex decomposable discrete sets with the same horizontal, vertical, and diagonal projections H, V, D, respectively.*

Unfortunately, Algorithm 4-RECDEC is not suitable for solving the corresponding consistency task, i.e., for determining the $\textsc{Consistency}(\mathcal{HV} \cap \mathcal{DEC},\ \mathcal{L}_4)$ task. It may happen the fact that the algorithm reconstructs an hv-convex set with the given projections despite the assumption that the components are uniquely determined is wrong and so property (α) is not satisfied. Clearly in this case the reconstructed set is not in $\mathcal{HV} \cap \mathcal{DEC}$. This 'problem' can be turned to our advantage to get a fast and accurate reconstruction heuristic (called Algorithm 4-RECHV) that works for a much broader subclass of hv-convexes than the decomposable subclass. We can modify Algorithm 4-RECDEC in such a way that it will be suitable for reconstructing hv-convex discrete sets that satisfy properties (β) and (γ) but not necessarily property (α).

To test the accuracy and speed of this heuristic we carried out a series of experiments. We generated 5 datasets, each of them containing 1000 hv-convex sets – which satisfied properties (β)

Table 2: Accuracy and average running time of Algorithm 4-RECHV on the test datasets

Test	#correct sol.	#incorrect sol.	#no sol.	time (s)
Test 1	939	14	47	0.600
Test 2	891	27	82	0.847
Test 3	851	41	108	2.322
Test 4	998	0	2	0.660
Test 5	994	0	6	5.676

and (γ) – that had k components of size $n \times n$ for some fixed k and n. For the 5 test datasets the k and n parameters had the following values. Test 1: $k = 10$, $n = 5$; Test 2: $k = 20$, $n = 5$; Test 3: $k = 30$, $n = 5$; Test 4: $k = 10$, $n = 10$; Test 5: $k = 20$, $n = 10$. Table 2 shows the average running times, and the number of correct and incorrect solutions for the 5 test datasets. From the test results we can deduct that Algorithm 4-RECHV has a good performance in terms of both quality and running time.

The concept of decomposability and the decomposition technique was described in [5; 8]. The applications of the technique for several decomposable classes can be found in [2; 5; 9]. The negative result concerning the ambiguity problem using three projections and the reconstruction heuristic came from results presented in [6].

Random Generation of hv-Convex Discrete Sets

In the last few years the class of hv-convex discrete sets has become one of the statistical indicators of newly developed exact or heuristical reconstruction algorithms which characterises the effectiveness (accuracy, speed, etc.) of a given technique. This means that the performance of the newly developed techniques are often tested on this class. Unhappily researchers had to acknowledge the fact that no method was known for generating hv-convex sets of given sizes from uniform random distributions, and hence no exact comparison of the techniques was possible. In the following we present an algorithm for generating hv-convex discrete sets of a moderate size from uniform random distributions and study the properties of uniformly generated sets which can affect the performance of certain reconstruction algorithms.

In [13] it was shown that the number $P_{m+1,n+1}$ of hv-convex polyominoes with a minimal bounding rectangle of size $(m+1) \times (n+1)$ is

$$P_{m+1,n+1} = \frac{m+n+mn}{m+n}\binom{2m+2n}{2m} - \frac{2mn}{m+n}\binom{m+n}{m}^2 . \qquad (4)$$

We will now consider a special class of hv-convex discrete sets (denoted by $\mathcal{S}'$), namely where the minimal bounding rectangles of the components are always connected to each other with their bottom right hand and upper left hand corners and there are no empty rows and columns in the discrete sets. It is not hard to see that the number $S'_{m,n}$ of discrete sets of $\mathcal{S}'$ with size $m \times n$ can be expressed by a simple recursive formula. It is

$$S'_{m,n} = P_{m,n} + \sum_{k<m,\ l<n} P_{k,l} \cdot S'_{m-k,n-l} . \qquad (5)$$

Using Equation (4) and the initial values $S'_{1,j} = P_{1,j} = 1$ $(j = 1,\dots,n)$ and $S'_{i,1} = P_{i,1} = 1$ $(i = 1,\dots,m)$ $S'_{m,n}$ can be calculated via a dynamic programming approach in $O(m^2n^2)$ time with $O(mn)$ memory requirements. From this, we now can describe the algorithm (called Algorithm GENHV-S') for generating hv-convex discrete sets of $\mathcal{S}'$ of a given size using a uniform random distribution. This algorithm uses the method of [16] to generate the components using uniform random distributions – knowing their minimal bounding rectangles.

Algorithm GENHV-S' for generating sets of $\mathcal{S}'$ using a uniform random distribution

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Input: The integers m and n.
Output: The hv-convex discrete set $F \in \mathcal{S}'$ of size $m \times n$.

Step 1 calculate $S'_{m,n}$;
Step 2 generate a number $r \in [1, S'_{m,n}]$ using a uniform random distribution;
Step 3 **if** ($r > P_{m,n}$)
  { $r = r - P_{m,n}$;
   for $k = 1$ **to** $m - 1$
    for $l = 1$ **to** $n - 1$
     if ($r > P_{k,l} \cdot S'_{m-k,n-l}$) $r = r - P_{k,l} \cdot S'_{m-k,n-l}$;
     else call Algorithm GENHV-S' with parameters $m - k$ and $n - l$; }
Step 4 generate the components using a uniform random distribution;

The above method can also be extended to special hv-convex discrete sets which may have empty rows or/and columns. This class will be denoted by $\mathcal{S}$. Denoting the number of discrete sets of $\mathcal{S}$ with size $m \times n$ by $S_{m,n}$, we get a formula similar to Equation (5). That is,

$$S_{m,n} = P_{m,n} + \sum_{k<m,\ l<n} P_{k,l} \cdot \Big(\sum_{i \le m-k,\ j \le n-l} S_{i,j} \Big) \ . \tag{6}$$

After, an algorithm similar to Algorithm GENHV-S' can be supplied to generate hv-convex discrete sets of given size from the $\mathcal{S}$ class based on a uniform random distribution (let us call this Algorithm GENHV-S). It needs, though, some rethinking to adapt Algorithm GENHV-S' (and Algorithm GENHV-S) to the whole class of hv-convexes. We will use the following lemma whose content is quite trivial.

Lemma 6 *Let F be an arbitrary hv-convex set that has $k \ge 2$ components. Then there is a uniquely determined set $F' \in \mathcal{S}$ and a uniquely determined permutation π of order k such that F and F' have the same components and if the SCDR of the l-th component of F' is $\{i_l,\dots,i'_l\} \times \{j_l,\dots,j'_l\}$ then the SCDR of the l-th component of F is $\{i_l,\dots,i'_l\} \times \{j_{\pi(l)},\dots,j'_{\pi(l)}\}$.*

We need to make some small changes in Algorithm GENHV-S' to achieve our goal. In fact, what we have to do is to have proper weights of the discrete sets of $\mathcal{S}'$ that can represent several hv-convex sets. These sets of $\mathcal{S}'$ are exactly the ones which consist of several components. Moreover, with the help of Lemma 6 it is also clear that each such set represents $k!$ sets if it has k components.

Now, the recursive formulas used in the generation will change in the following way. Let $HV'^{(t)}_{m,n}$ denote the number of arbitrary hv-convex discrete sets with nonempty rows and columns that have minimal bounding rectangle of size $m \times n$ and exactly t components. Then $HV'^{(t)}_{i,j} = 0$ if $i < t$

or $j < t$, and $HV'^{(1)}_{i,j} = P_{i,j}$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Finally, based on similar observations as the formula of (5) for $t > 1$ the following recursion holds

$$HV'^{(t)}_{m,n} = \sum_{k<m,\ l<n} P_{k,l} \cdot HV'^{(t-1)}_{m-k,n-l} \cdot t \ . \tag{7}$$

Here, the factor t decreases by one at each step as we go deeper and deeper in the recursion. Eventually this will yield a factor of $t!$ for a set that consists of t components – which is exactly the weight we need. Afterwards, with a simple calculation we find that the total number $HV'_{m,n}$ of arbitrary hv-convex discrete sets of size $m \times n$ with nonempty rows and columns is

$$HV'_{m,n} = \sum_{t=1}^{\min\{m,n\}} HV'^{(t)}_{m,n} \ . \tag{8}$$

Now, in an analogous way to Algorithm GENHV-S' shall describe the generation method called Algorithm GENHV' for the whole class of hv-convexes with nonempty rows and columns, also taking into account what Lemma 6 says – that the components need to be permuted by a randomly chosen permutation. We should also add that with the help of equations (6) and (7) we can develop Algorithm GENHV in a straightforward way for generating arbitrary hv-convex sets – perhaps with empty rows or/and columns – using a uniform random distribution. After, the total number of hv-convex discrete sets can be calculated by a formula similar to (8).

Our generation methods developed allow us to examine some important properties of hv-convex discrete sets. In order to get some statistical information about them we generated test datasets with all four generation algorithms. Each set of test data consisted of 1000 hv-convex discrete sets of the same size generated from the given class using a uniform random distribution.

Our first investigation focused on the number of hv-convex discrete sets in the classes studied. Introducing the notations $\mathcal{P}$ for the class of hv-convex polyominoes and $\mathcal{HV}'$ for the class of hv-convex discrete sets with non-empty rows and columns the following inclusions are trivial: $\mathcal{P} \subset \mathcal{S}' \subset \mathcal{S} \subset \mathcal{HV}$ and $\mathcal{P} \subset \mathcal{S}' \subset \mathcal{HV}' \subset \mathcal{HV}$. Knowing these relations and calculating the number of elements in these classes with bounding rectangles of semi-perimeter n for a fixed n, we can describe the relative cardinality of the classes examined. With this information we can, for example, address questions concerning the relative occurrence of certain hv-convex discrete sets and calculate the probability that an hv-convex discrete set chosen from a uniform random distribution has some special properties which can facilitate the reconstruction task.

The second experiment examines the number of components of an hv-convex set. This piece of information is also very useful when reconstructing images like these. As regards the $\mathcal{S}'$ and $\mathcal{S}$ classes we found that the expected number of components of the special hv-convex discrete set that was generated using a uniform random distribution can also be estimated in advance by knowing just the size of the discrete set. This could be quite useful in the reconstruction task. In more detail, let $E_{\mathcal{C}}(n)$ and $D^2_{\mathcal{C}}(n)$ respectively denote the expected number and the variance of the components of a discrete set of size $n \times n$ generated from a given $\mathcal{C}$ class using a uniform random distribution. Then for sets of size larger than about 100×100 a good estimation can be given by $E_{\mathcal{S}'}(n) \approx 0.075n$ and $D^2_{\mathcal{S}'}(n) \approx 0.04n$ in the $\mathcal{S}'$ class, and $E_{\mathcal{S}}(n) \approx 0.100n$ and $D^2_{\mathcal{S}}(n) \approx 0.06n$ in the $\mathcal{S}$ class. In addition, in both classes $\mathcal{S}'$ and $\mathcal{S}$ and for each size of sets the number of components follows a normal-like distribution with expectation value $E_{\mathcal{S}'}(n)$ and with variance $D^2_{\mathcal{S}'}(n)$ (and with expectation value

$E_{\mathcal{S}}(n)$ and with variance $D^2_{\mathcal{S}}(n)$ in the $\mathcal{S}$ class). In order to verify this we decided to generate two more test sets made up of 1000 uniformly chosen discrete sets of sizes 200×200 and 500×500 from the $\mathcal{S}'$ class. Figure 1 shows the differences between the test results and the normal distributions with the estimated parameters.

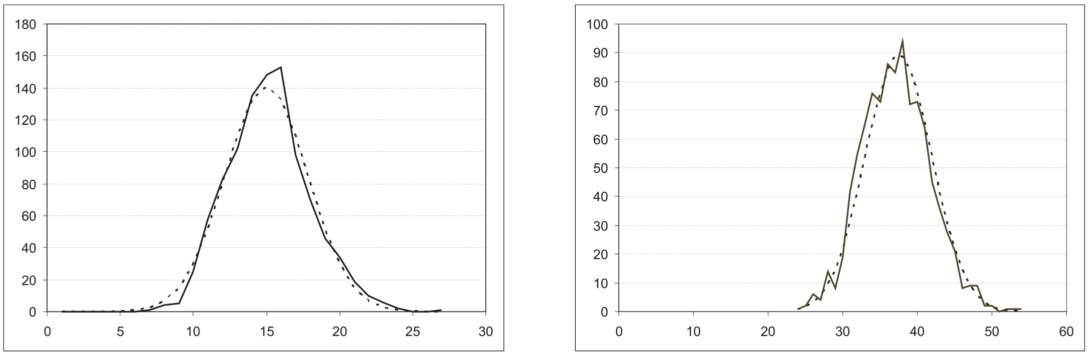

Figure 1: The distribution of the number of components in the test dataset (solid lines) and the corresponding normal distribution (dashed lines) for sets of $\mathcal{S}'$ of sizes 200×200 (left) and 500×500 (right)

With the aid of the formulas (7) and (8) (and with similar formulas in the classes $\mathcal{S}$, $\mathcal{S}'$, and $\mathcal{HV}$) it is also possible to describe the true distribution of the number of components of the generated hv-convex discrete set of the $\mathcal{HV}'$ class since, in this case, we can enumerate the discrete sets of a given class that has k components. Table 3 lists the expectation values and the variances of the variables that represent the number of components of a discrete set generated using a uniform random distribution from the $\mathcal{HV}'$ and $\mathcal{HV}$ classes when the size of the minimal bounding rectangle is $n \times n$ for a fixed $n \in \mathbb{N}$. For the $\mathcal{HV}$ class the statistics for sets of size larger than 80×80 could not be determined due to the huge time complexity of the generation algorithm. This shows the (perhaps single) drawback of the our generation method, namely that it is applicable for generating general hv-convex discrete sets of moderate size only.

Table 3: The expectation value $E_{\mathcal{HV}'}(n)$ $(E_{\mathcal{HV}}(n))$ and the variance $D^2_{\mathcal{HV}'}(n)$ $(D^2_{\mathcal{HV}}(n))$ of the components of a set with a minimal bounding rectangle of size $n \times n$ in the $\mathcal{HV}'$ $(\mathcal{HV})$ class. The values have been rounded to 5 digits

n	$E_{HV'}(n)$	$D^2_{HV'}(n)$	$E_{HV}(n)$	$D^2_{HV}(n)$
20	6.53981	9.84446	9.03570	8.12406
40	26.33821	16.00766	26.11090	9.54114
60	46.30283	12.92260	43.68220	10.00145
80	65.70631	12.05665	61.49588	10.72577
100	84.99456	11.80716	–	–
200	181.53870	12.45513	–	–

Despite this, statistics about the expected number of components can be especially useful in the reconstruction task. It tells us something about the discrete set to be reconstructed before we attempt to reconstruct it. Thus, such statistics opens the way for designing reconstruction algorithms in the future that exploit information known beforehand about the expected number of the components.

The generation method described in this section and some of the statistics presented here can be found in [7].

Conclusions

The use of geometrical priors in the reconstruction of binary images (or, equivalently, discrete sets) from their projections is an effective tool for overcoming ambiguity and intractability problems which arise from using a limited number of available projections. Connectedness, convexity, and directedness are the most frequently used geometrical properties to facilitate the reconstruction process. If the discrete set to be reconstructed has only one of these properties then the reconstruction usually does not become easier. In this dissertation we gave a far from exhaustive description of the hybrid combinations of the above key properties that can guarantee fast and rare ambiguous reconstruction.

We found that the assumption that the discrete set to be reconstructed is *directed and convex* along certain directions can yield a unique reconstruction using the horizontal and vertical projections in polynomial time, but this result is very sensitive in the sense that even small changes in the direction of convexity can yield an enormous number of solutions with the same two projections. Combinations of variants of *connectedness and convexity* properties can also guarantee polynomial-time reconstruction from the horizontal and vertical projections. Here, we get some good theoretical results about the number of the possible solutions and some good information about the reconstruction complexity.

In the case if four projections are available we introduced the concept of *decomposability* and showed that every decomposable discrete set can be reconstructed from the horizontal, vertical, diagonal and antidiagonal projections in polynomial time. This algorithm is an especially important result of the thesis since up to now very few reconstruction algorithms are known that use four projections. In addition, it algorithm can serve a basis for some reconstruction heuristics that use four projections as well.

Besides this, we solved the problem of generating hv-convex discrete sets using a uniform random distribution. The method presented can be adapted to many classes. With the aid of this method we also present some statistics about hv-convex discrete sets which are of help in analysing and designing efficient reconstruction algorithms.

The theoretical results given in this dissertation are interesting on their own right as well, but the insights we gain into the mathematical behaviour of binary tomography could be more important. Of course, it is necessary to understand the theory of binary tomography for designing efficient reconstruction algorithms that can be applied in practice as well.

Summary of the Author's Contributions

In the following we summarize the results of the author by arranging them into six thesis points.

I.) Horizontally or vertically convex NE-directed polyominoes can be reconstructed from their horizontal and vertical projections uniquely in polynomial time. The author investigated how varying the direction of convexity influences the above result. He found that the above result can be extended to diagonally convex NE-directed polyominoes as well, but assuming convexity along any other directions can yield an exponentially large number of solutions.

	[1]	[2]	[4]	[5]	[6]	[7]
I.	•					
II.		•				
III.		•	•			
IV.		•	•	•		
V.					•	
VI.						•

Table 4: The connection between the thesis points and the corresponding publications

II.) The author developed a fast algorithm for reconstructing Q-convex discrete sets which have at least two components using two projections. The author's algorithm has a worst case time complexity of $O(mn \cdot \min\{m^2, n^2\})$ and it can locate all the solutions of a given reconstruction task.

III.) The author showed that the class of hv-convex 8-connected discrete sets form a subclass of the class of Q-convexes. He compared his algorithm on the class of hv-convex 8-connected sets to previously published ones and found that the new algorithm outperforms the former ones from the viewpoint of the worst case and the viewpoint of the average execution time complexity. He also showed that for Q-convex but not 8-connected sets his algorithm can be speeded up to having a worst case time complexity of $O(mn)$, and in this case the number of possible solutions of the same reconstruction problem is at most two.

IV.) The author introduced the class of decomposable discrete sets and presented a polynomial-time reconstruction algorithm for this class using four projections. He investigated the relationship between the classes of decomposables and Q-convexes and demonstrated its consequences for the reconstruction task complexity for some well-known classes when four projections are available.

V.) The author investigated the possibility of extending the decomposition technique to the class of hv-convexes. He presented a fast and accurate heuristic reconstruction algorithm for hv-convex sets with decomposable configurations.

VI.) The efficiency of recently developed exact or heuristical reconstruction algorithms are often tested on the class of hv-convexes. The author described a method for generating elements of this class using a uniform random distribution where an exact comparison of several reconstruction algorithms can be made from the viewpoint of average execution time. The method can readily be extended to other classes of discrete sets which have disjoint components. Using this method the author presented statistics on properties of hv-convex sets that can affect the performance of reconstruction algorithms.

The research presented in the dissertation resulted in several publications. Table 4 summarizes which publication covers which item of the thesis points.

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