Tételek a konvex geometria és az analízis határán

Doktori értekezés

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Előszó

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A disszertáció szakmai része – a magyar összefoglalót kivéve – angol nyelven íródott. Mivel az elmúlt 50 évben ez a nyelv gyakorlatilag egyeduralkodóvá vált a matematikában, és túlzás nélkül állíthatjuk, hogy minden matematikus legalább szakmai szinten beszéli, ez remélhetőleg nem jelent korlátozó tényezőt.

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Chapter 1
Transversals of unit balls

In the present chapter we prove the following statement, which belongs to geometric transversal theory.

Theorem 1.1. *Let $d \geqslant 2$, and $\mathcal{F}$ be a family of unit balls in $\mathbb{R}^d$ with the property that the mutual distances of the centres are at least $2\sqrt{2+\sqrt{2}}$. If every at most d^2 members of $\mathcal{F}$ have a common line transversal, then all members do.*

This is a joint result with András Bezdek and Ferenc Fodor, which was published in [ABF06]. Progress did not halt then, and there have been further developments related to Theorem 1.1 since the publication of [ABF06]. These, along with the detailed history of the subject, are presented in Section 1.1.

The proof of Theorem 1.1 is based on the following fact. For a given collection of unit balls in $\mathbb{R}^d$ satisfying the above distance condition on the pairwise distances of the centres, consider the set of direction vectors of the common transversals which intersect the balls in a fixed order. The resulting set, which is a subset of S^{d-1}, is strictly spherically convex. This property is proved in Section 1.2. Then, in Section 1.3, orderings of the balls induced by common transversals are considered; these are called geometric permutations. Using properties of them, Theorem 1.1 is proved in Section 1.4 by invoking a version of Helly's theorem.

1.1 History

First, we give a short account of the quite extended history of the problem. For further details, consult the survey articles [Eck93], [DGK63], [GPW91] and [Wen99].

A line l is a *transversal* to a family $\mathcal{F}$ of sets if l intersects every element of $\mathcal{F}$. If there is a line that intersects every member of the family $\mathcal{F}$, then we say that $\mathcal{F}$ has the property T. If every k or fewer members of $\mathcal{F}$ have a transversal then $\mathcal{F}$ has property $T(k)$.

In 1958, Grünbaum [Grü58] conjectured that for a family of pairwise disjoint translates of a convex disk $T(5) \Rightarrow T$. The special case of circular disks was settled by Danzer [Dan57]. The conjecture was proved in full generality by Tverberg [Tve89] in 1989.

Higher dimensional generalisations for families of balls were initiated by Hadwiger [Had56]. He stated that for any family of *thinly distributed* balls in $\mathbb{R}^d$ the property $T(d^2)$ implies T. A family of balls is *thinly distributed* if the distance between the centers of any two balls is at least twice the the sum of their radii. In 1960, Grünbaum [Grü60] improved the d^2 in Hadwiger's statement to $2d-1$ using the Topological Helly Theorem.

Imposing such a condition on the distances between the centres is natural. To see this, consider the following example. Take a regular n-gon of unit side length in the plane and place circular disks of diameter $1/2$ centred at the vertices. Enlarge the disks from their centres with the same factor λ. There is a minimal λ for which every $n-1$ enlarged disks have a common transversal but there is no transversal to all n disks. It is easy to see that this minimal λ is equal to the minimal width of an $(n-1)$-gon obtained by dropping a vertex of a regular n-gon of unit side length. In this configuration, $T(n-1)$ holds, but T does not. Therefore, the minimal k for which $T(k) \Rightarrow T$, is not independent of the minimum pairwise distance of the centres of the disks. Investigations in this direction were initiated by Heppes. Recently, K. Bezdek, Bisztriczky, Csikós, and Heppes [BBCH06] proved new results in this direction.

In 2003, Holmsen, Katchalski and Lewis [HKL03] proved that there exists a positive integer $n_0 \leqslant 46$ such that $T(n_0)$ implies T for any family of pairwise disjoint unit balls in $\mathbb{R}^3$. This bound was improved by Cheong, Goaoc and Holmsen [CGH05] to 11. On the other hand, a result of Holmsen and Matoušek [HM04] states that there is no such Helly-number if one considers families of pairwise disjoint translates of an arbitrary convex body in $\mathbb{R}^3$.

Our result works with a condition on the distances between the centres of the balls, which is weaker than Hadwiger's thin distribution condition. Theorem 1.1 generalises the main result of [HKL03], and it also strengthens Hadwiger's theorem [Had56] for congruent balls. The main tool of the proof is the convexity of the cone of transversal directions, that is shown in Section 1.2.

The method presented here has been pushed further in [CGHP08], where the convexity of the cone of transversal directions was showed without the extra distance condition - in fact, this property was established under a condition which is even weaker than disjointness. Thus, the authors of [CGHP08] derived that the components of the set of transversal directions are contractible. Moreover, in [CGN05] it has been shown that any family of at least 9 disjoint unit balls admits at most two geometric permutations. Using these facts and the Topological Helly theorem, a linear bound can be obtained instead of the quadratic one. This has been accomplished in [CGHP08], where the authors proved that for any system of disjoint unit balls in $\mathbb{R}^d$, $T(4d-1)$ implies T. This is currently the strongest result regarding this problem. For the latest developments, see the surveys of Goaoc [Goa09] and Holmsen [Hol08].

Throughout the chapter, $\mathcal{F}_d$ will denote a family of d-dimensional unit balls with the property that the distances of the centres of any two of them are at least $2\sqrt{2+\sqrt{2}}$ (we shall abbreviate this property as the "distance condition").

1.2 Convexity of the cone of transversal directions

Let $B_1, \ldots, B_m$ be disjoint unit balls in $\mathbb{R}^d$. Consider the set of all directed lines intersecting $B_1, \ldots, B_m$ in this order, and denote the set of unit direction vectors of these lines by $\mathcal{K}(B_1, \ldots, B_m)$. Then $\mathcal{K}(B_1, \ldots, B_m) \subset S^{d-1}$. The goal of the section is to verify the following statement, which is based upon Lemma 2.1 in [HKL03].

Theorem 1.2. *Let $\mathcal{F}_d = \{B_1, \ldots, B_m\}$ be a family of unit balls satisfying the distance condition. Then $\mathcal{K}(B_1, \ldots, B_m)$ is convex.*

Before starting the proof, we present an auxiliary lemma, that also serves as a motivation for Theorem 1.2.

Lemma 1.3. *Let $\{K_1, \ldots, K_m\}$ be a family of disjoint disks in the plane. If there exists a direction α such that for every two disks K_i, K_j, $1 \leqslant i < j \leqslant m$, there is a transversal of direction α which intersects K_i first and K_j second, then there is a transversal of direction α that intersects $K_1, \ldots, K_m$ in this order.*

Proof. Let l be a line perpendicular to the direction α, and take the orthogonal projection of the family $\{K_1, \ldots, K_m\}$ to l. The images of the disks will be segments, and from the assumptions we know that they are pairwise intersecting. Therefore we can apply the Helly theorem in dimension 1 and obtain that the intersection of all of the segments is not empty, say, it contains the point N. Then the line of direction α through N is a suitable transversal. □

We note that Lemma 1.3 proves the assertion of Theorem 1.2 in dimension 2. It is obvious that

$$\mathcal{K}(B_1, \ldots, B_m) \subseteq \bigcap_{1 \leqslant i < j \leqslant m} \mathcal{K}(B_i, B_j)$$

and the Lemma shows that equality holds. It is clear that no $\mathcal{K}(B_i, B_j)$ can contain antipodal points of $\mathcal{S}^1$. If we pick two points of $\mathcal{K}(B_1, \ldots, B_m)$, then all of the sets $\mathcal{K}(B_i,\ B_j)$ must contain one of the two arcs between them in $\mathcal{S}^1$, and our previous statement implies that this arc can only be the shorter one. Therefore, $\mathcal{K}(B_1, \ldots, B_m)$ contains the small arc between any of its two points, hence it is convex.

Proof of Theorem 1.2. First we prove the statement for the case $m = 2$. By definition of convexity we have to show that for any two directions $\alpha_1, \alpha_2 \in \mathcal{K}(B_1, B_2)$, every normed linear combination of the form

$$\frac{\alpha(\lambda)}{||\alpha(\lambda)||} \ , \quad \text{where} \quad \alpha(\lambda) = \lambda\alpha_1 + (1-\lambda)\alpha_2 \ , \ 0 \leqslant \lambda \leqslant 1 \tag{1.1}$$

is in $\mathcal{K}(B_1, B_2)$ as well. Let P denote the plane generated by α_1 and α_2, and let $P^\perp$ denote the orthogonal complement of P in $\mathbb{R}^d$. Let $H(z) = P + z$ where $z \in P^\perp$. Then for every $\alpha \in P \cap \mathcal{K}(B_1, B_2)$ there exists an appropriate transversal to B_1, B_2 parallel to α, therefore there exists a $z(\alpha) \in P^\perp$ such that we can find an appropriate transversal in $H(z(\alpha))$ with direction α (of course, $z(\alpha)$ is not necessarily unique). Fix such a $z(\alpha)$ for every $\alpha \in P \cap \mathcal{K}(B_1, B_2)$.

We consider two cases depending on $z(\alpha_1)$ and $z(\alpha_2)$. First, assume that they are equal. Then there exists a plane G and two transversals l_1 and l_2 in G such that the direction of l_1 is α_1 and the direction of l_2 is α_2. We show that for every direction determined by a combination of the form (1.1) there exists a transversal l in G. Let $K_1 = B_1 \cap G$, $K_2 = B_2 \cap G$. Then K_1, K_2 are circular disks in G and the transversals to B_1 and B_2 in G are exactly the transversals to K_1 and K_2 in the plane G and it is clear that $\mathcal{K}(K_1, K_2)$ is convex.

Second, suppose that $z(\alpha_1) \neq z(\alpha_2)$. Let m be the segment $z(\alpha_1)z(\alpha_2)$. Since $m \subset P^\perp$, m is orthogonal to P, and the subspace T generated by m and P is of dimension 3. We prove that for every combination of the form (1.1), there exists a $z \in m$ such that in $H(z)$ we can find an appropriate transversal. Let $G_1 = B_1 \cap T$, $G_2 = B_2 \cap T$ be 3-dimensional balls. Note that G_1 and G_2 cannot degenerate to points. Then for every $z \in m$ the transversals in $H(z)$ which intersect B_1 and B_2 in this order are exactly the transversals which intersect G_1 and G_2 in this order, therefore we have to handle a 3-dimensional problem.

By scaling and symmetry we may suppose that the radius of G_1 is 1 and the radius of G_2 is $r \leqslant 1$. Note that since the unscaled radii of G_1 and G_2 are at most 1, the scaled distances are at least as large as the unscaled ones. Choose the coordinate system in T such that the z-axis contains m, and the centres of G_1 and G_2 are $(0, 0, 0)$ and $(d, 0, b)$, respectively, with $d \geqslant 0$ and $b \geqslant 0$. For $b - r \leqslant z \leqslant \min(b + r, 1)$ the intersections $H(z) \cap G_1$ and $H(z) \cap G_2$ are two, possibly degenerated, disks of radius $R_1(z)$ and $R_2(z)$, respectively, where

$$R_1(z) = \sqrt{1 - z^2}\ ,\ R_2(z) = \sqrt{r^2 - (b - z)^2}.$$

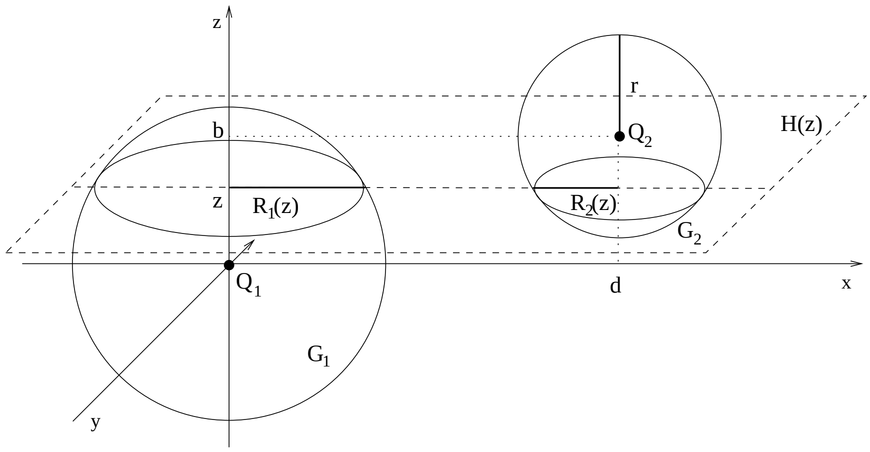

Figure 1.1: The intersection of the d-dimensional balls and $\mathcal{T}$

Consider the transversals l in $H(z)$ for $b - r \leqslant z \leqslant \min(b + r, 1)$ intersecting G_1 before G_2. To each l we assign the angle $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ between the x-axis and l. For every z we get a possible minimal and maximal angle between the x-axis and an appropriate transversal in $H(z)$, say $\varphi_{\min}(z)$ and $\varphi_{\max}(z)$, respectively. By symmetry, $\varphi_{\max}(z) = -\varphi_{\min}(z)$. Denote $\varphi_{\max}(z)$ by simply $\varphi(z)$. If z traverses from $b - r$ to $\min(b + r, 1)$, then the possible (z, φ) belonging to the order respecting transversals to G_1 and G_2 form a bounded region $W(B_1, B_2)$ in the (z, φ) plane. It suffices to show that this region is convex. We shall prove that the upper boundary of $W(B_1, B_2)$ is concave:

The function $\varphi(z)$ is concave for $b - r \leqslant z \leqslant \min(b + r, 1)$. (*)

For a given z, we easily obtain that

$$\varphi(z) = \arcsin(f(z)),\ \text{where}\ \ f(z) = \frac{R_1(z) + R_2(z)}{d}.$$

We shall compute the second derivative of $\varphi(z)$, and verify that $\varphi''(z) \leqslant 0$ for $b - r \leqslant z \leqslant \min(b + r, 1)$. Easy computations show that

$$f'(z) = -\frac{1}{d}\left(\frac{z}{R_1(z)} + \frac{z-b}{R_2(z)}\right),$$

$$f''(z) = -\frac{1}{d}\left(\frac{1}{(R_1(z))^3} + \frac{r^2}{(R_2(z))^3}\right),$$

$$\varphi''(z) = \frac{f''(z)\left(1 - (f(z))^2\right) + f(z)(f'(z))^2}{\left(1 - (f(z))^2\right)^{3/2}}.$$

Hence it suffices to show that in the given interval

$$g(z) = f''(z)\left(1 - (f(z))^2\right) + f(z)(f'(z))^2 \leqslant 0$$

holds.

By substituting the formulae for $f(z)$, $f'(z)$, and $f''(z)$ in $g(z)$, the desired inequality turns into

$$-\frac{1}{d}\left(\frac{1}{(R_1(z))^3} + \frac{r^2}{(R_2(z))^3}\right)\left(1 - \frac{(R_1(z) + R_2(z))^2}{d^2}\right) \leqslant -\frac{R_1(z) + R_2(z)}{d} \cdot \frac{1}{d^2} \cdot \left(\frac{z}{R_1(z)} + \frac{z-b}{R_2(z)}\right)^2.$$

Multiplying both sides by $-d^3 R_1^3(z) R_2^3(z)$ we obtain

$$\left(R_2^3(z) + r^2 R_1^3(z)\right)\left(d^2 - (R_1(z) + R_2(z))^2\right) \geqslant R_1(z)R_2(z) \cdot \cdot (R_1(z) + R_2(z))\left(z^2 R_2^2(z) + 2z(z-b)R_1(z)R_2(z) + (z-b)^2 R_1^2(z)\right). \quad (1.2)$$

Since $r \leqslant 1$, we have that

$$R_2^3(z) + r^2 R_1^3(z) \geqslant (R_2(z) + rR_1(z))\left(R_2^2(z) - rR_1(z)R_2(z) + r^2 R_1^2(z)\right). \quad (1.3)$$

Furthermore, $|z| \leqslant 1$ and $|z - b| \leqslant r$ yield

$$\left(z^2 R_2^2(z) + 2z(z-b)R_1(z)R_2(z) + (z-b)^2 R_1^2(z)\right) \leqslant \left(R_2^2(z) + 2rR_1(z)R_2(z) + r^2 R_1^2(z)\right). \quad (1.4)$$

We shall use the following elementary inequality : if $a, b \in \mathbb{R}$ then

$$a^2 + 2ab + b^2 \leqslant 4(a^2 - ab + b^2). \tag{1.5}$$

With (1.3), (1.4) and (1.5), we obtain that the following inequality implies (1.2):

$$(R_2(z) + rR_1(z))\left(d^2 - (R_1(z) + R_2(z))^2\right) \;\geqslant\; 4R_1(z)R_2(z)\left(R_1(z) + R_2(z)\right). \tag{1.6}$$

Using $r \leqslant 1$ and dividing both sides of (1.6) by $r(R_1(z)+R_2(z))$ yield that the following inequality implies (1.6):

$$d^2 \geqslant 4\frac{R_1(z)R_2(z)}{r} + (R_1(z) + R_2(z))^2. \tag{1.7}$$

Since $R_1(z) \leqslant 1$ and $R_2(z) \leqslant r$, the right hand side of (1.7) is at most 8. We will use the following proposition to finish the proof of (*).

The Euclidean distance between the centres of G_1 and G_2 is at least $2\sqrt{1+\sqrt{2}}$. (**)

Clearly, (**) is true in 2 and 3 dimensions. Consider the case $n \geqslant 4$. Let O_1, O_2, Q_1, Q_2 denote the centres of B_1, B_2, G_1, G_2, respectively. The segments O_1Q_1 and O_2Q_2 are orthogonal to T, hence they are perpendicular to the segment Q_1Q_2. Furthermore, $|O_1Q_1| \leqslant 1$, $|O_2Q_2| \leqslant 1$, and $|O_1O_2| \geqslant 2\sqrt{2+\sqrt{2}}$. Therefore we obtain that $|Q_1Q_2| \geqslant 2\sqrt{1+\sqrt{2}}$.

Notice that the scaled distance of the centres in T is at least as big as the Euclidean distance.

We divide the rest of argument proving (*) into two cases.

Case 1. $b \leqslant 1$.

Since the distance between the centres of G_1 and G_2 is $t = \sqrt{b^2 + d^2}$, we obtain that $d^2 \geqslant 3 + 4\sqrt{2} > 8.6$. It has already been shown that the right hand side of (1.7) is at most 8, which yields that the inequality (1.7) holds.

Case 2. $b > 1$.

Now, we are going to estimate $R_1(z)$ and $R_2(z)$. Since $[b-r, 1]$ is contained in $[b-1, 1]$, the following inequalities hold: $R_1(z) \leqslant \sqrt{1-(b-1)^2}$ and $R_2(z) \leqslant \sqrt{1-(b-1)^2}$. Furthermore, $R_2(z) \leqslant r$. Using this information, we may estimate the right hand side of (1.7) by

$$4\sqrt{1-(b-1)^2} + 4(1-(b-1)^2) \geqslant 4\frac{R_1(z)R_2(z)}{r} + (R_1(z) + R_2(z))^2. \tag{1.8}$$

Using $d^2 = t^2 - b^2$, proposition (**) and formula (1.8), we obtain that it suffices to show the following inequality.

$$4(1+\sqrt{2}) \geqslant b^2 + 4\sqrt{1-(b-1)^2} + 4(1-(b-1)^2). \tag{1.9}$$

Easy computation shows that the maximum of the right hand side of (1.9) is smaller than 9.3 in the interval $1 < b \leqslant 2$. Therefore the inequality (1.9) holds, and this finishes the proof of (*) and proves Theorem 1.2 for $m = 2$.

Next, we prove Theorem 1.2 for $m \geqslant 3$. Let $\alpha_1, \alpha_2 \in \mathcal{K}(B_1, \ldots, B_m)$, and again, consider a combination of the form (1.1). We shall use the same notations as in the proof of the case $m = 2$. Suppose first that $z(\alpha_1) = z(\alpha_2) = z$. Then in the plane $H(z)$ the transversals to the family $B_1, \ldots, B_m$ will be exactly the transversals to the family $K_1, \ldots, K_m$, the family of the intersections of the balls and the plane $H(z)$. Lemma 1.3 implies that Theorem 1.2 holds in the plane, and thus it verifies the statement in the case $z(\alpha_1) = z(\alpha_2) = z$.

Assume now that $z(\alpha_1) \neq z(\alpha_2)$. Notice that for every pair of balls B_i, B_j we use a different coordinate system and metric in T. However, the z axis is the same because it is determined by the segment m. Changing the directions of the x and y axes results in a translation parallel to the α axis of $W(B_i, B_j)$ in the (z, φ)-plane, and changing the position of the origin on z results in a translation in the z direction. Reversing the direction of any of the axes results in an axial symmetry of $W(B_i, B_j)$. Scaling by a positive factor scales $W(B_i, B_j)$ along the z axis. None of these transformations changes the convexity of $W(B_i, B_j)$.

Now, fix a coordinate system in T among the given ones, and consider all $W(B_i, B_j)$ regions in this frame. Lemma 1.3 implies that

$$W(B_1, \ldots, B_m) = \bigcap_{1 \leqslant i < j \leqslant m} W(B_i, B_j).$$

We show that $\mathcal{K}(B_i, B_j)$ cannot contain antipodal points of $\mathcal{S}^{d-1}$. On the contrary, assume that there exists a direction $\alpha \in \mathcal{S}^{d-1}$ with the property that there are lines l_1 and l_2 of direction α and $-\alpha$, respectively, intersecting B_1 and B_2 in this order. This means that we can translate B_2 along l_1 to infinity in the direction α without crossing B_1. The same is true for the direction $-\alpha$ with the translation along l_2, hence there is no transversal parallel to α intersecting B_1 and B_2 , a contradiction.

Since $\mathcal{K}(B_i, B_j)$ cannot contain antipodal points, $\mathcal{K}(B_1, \ldots, B_m)$ must contain the (unique) geodesic between any two points of it, hence it is convex. $\square$

1.3 Geometric permutations

Let $\mathcal{F}$ be a family of pairwise disjoint convex bodies in $\mathbb{R}^d$, and let l be a transversal to $\mathcal{F}$. Then l induces two opposite orderings of the members of $\mathcal{F}$. These orders are essentially the same, and together they are called a *geometric permutation* of $\mathcal{F}$. We will consider the possible geometric permutations of the family $\mathcal{F}_d$. We may suppose that in $\mathcal{F}_n = \{B_1, \ldots, B_m\}$ the balls B' and B'' have the largest distance. Every geometric permutation is induced by the family $\mathcal{D}$ of directed transversals to $\mathcal{F}_d$ intersecting B' and B'' in this order. We say that a pair of balls B_i, B_j (where $1 \leqslant i < j \leqslant m$) forms a *switched pair* in $\mathcal{F}_d$ if there exist $l, l' \in \mathcal{D}$ that meet B_i, B_j in different orders. If $\mathcal{F}_d$ admits at least one geometric permutation and there are no switched pairs in $\mathcal{F}_d$, then clearly $\mathcal{F}_d$ admits exactly one geometric permutation. We will show that this is indeed the case.

LEMMA 1.4. *Let $\mathcal{F}_d$ be a family of unit balls satisfying the distance condition. Then $\mathcal{F}_d$ admits at most one geometric permutation.*

PROOF. Assume that there is a switched pair in $\mathcal{F}_d$, say, the pair B_1, B_2. Let O_1 and O_2 denote the centres of B_1 and B_2, respectively. There exist directed transversals l_1 and l_2 in $\mathcal{D}$ with the property that l_1 meets B_1 first and then B_2, while l_2 intersects B_2 first and B_1 second. Let P be a (two-dimensional) plane parallel to l_1 and l_2 in $\mathbb{R}^d$. We use the same notation as in the proof of Theorem 1.2. $P^\perp$ will denote the orthogonal complement of P in $\mathbb{R}^d$, and $H(z) = P + z$, where $z \in P^\perp$. For $k = 1, 2$ the intersection of B_k and $H(z)$ is a disk $K_k(z)$ with centre $C_k(z)$. Choose the directed line s in $\mathbb{R}^d$ such that the segments $C_1(z)C_2(z)$ are all parallel to it. Hence the angle between l_k and the directed line determined by $C_1(z), C_2(z)$ is equal to the angle between l_k and s. Proposition (**) yields that the distance between $C_1(z)$ and $C_2(z)$ is at least $2\sqrt[4]{2}$ (this quantity was denoted by d in the proof of Theorem 1.2). Since the radii of $K_1(z)$ and $K_2(z)$ are at most 1, the angle between the directed lines l_1 and s is at most $\arcsin(2/(2\sqrt[4]{2}))$. We obtain in the same way that the same holds for the angle between the directed lines l_2 and $-s$. Thus, the angle between the two transversals l_1 and l_2 is at least $\pi - 2\arcsin(1/\sqrt[4]{2})$.

On the other hand, both the transversals l_1 and l_2 intersect B' and B'' in this order. Since the distance of the centres of B' and B'' is at least $2\sqrt{2+\sqrt{2}}$, and their radius is 1, the angle between l_1 and l_2 is at most $2\arcsin(2/(2\sqrt{2+\sqrt{2}})) \sim 1.1437$.

It is easy to see that the upper and lower bounds for the angle between l_1 and l_2 cannot be sharp at the same time. Furthermore,

$$\arcsin(1/\sqrt[4]{2}) + \arcsin(1/\sqrt{2+\sqrt{2}}) = \pi/2,$$

since $1/\sqrt{2} + 1/(2+\sqrt{2}) = 1$. Thus, $\pi - 2\arcsin(1/\sqrt[4]{2}) \geqslant 2\arcsin(1/\sqrt{2+\sqrt{2}})$, a contradiction.

□

The following lemma will directly lead to Theorem 1.1.

LEMMA 1.5. *Suppose that the family $\mathcal{F}_n = \{B_1, \ldots, B_m\}$ has the property $T(d)$, where $d \geqslant 4$. Then there exists a linear ordering $\prec$ of $\{B_1, \ldots, B_m\}$ with the property that for every d–element subset $\mathcal{G} \subseteq \mathcal{F}_n$ there exists a transversal to $\mathcal{G}$ intersecting the elements of $\mathcal{G}$ compatible with the linear ordering $\prec$.*

PROOF. We will prove that there exists a geometric permutation of $\mathcal{F}_d$ with the desired property by induction on m .

If $m = d$ then there exists a transversal to $\mathcal{F}_d$, and we can choose the geometric permutation induced by it.

Suppose now that the claim holds for $m = k$, and we shall prove it for $m = k+1$. For every $2 \leqslant i \leqslant k$ the set $\{B_1, \ldots, B_{k+1}\} \setminus \{B_i\}$ has a geometric permutation with the desired property. For every such set consider the induced linear ordering $\prec_i$ for which $B_1 \prec_i B_{k+1}$ holds. If for any pair of balls B_r, B_s with $2 \leqslant r < s \leqslant k$ the orderings $\prec_i$, $2 \leqslant i \leqslant k$, $i \neq r, s$ agree, then we can uniquely extend the orderings to the whole family $\{B_1, \ldots, B_{k+1}\}$, and clearly, the extended ordering will have the desired property. To check this, suppose that there is a pair of balls B_i, B_j with $2 \leqslant i < j \leqslant k$ and a pair of orderings $\prec_r$, $\prec_s$ with $2 \leqslant r < s \leqslant k$, $\{r, s\} \cap \{i, j\} = \emptyset$ such that $B_i \prec_r B_j$ and $B_j \prec_s B_j$. Then the family $\{B_1, B_i, B_j, B_{k+1}\}$ admits two different geometric permutations, which is in contradiction to Lemma 1.4. Hence we obtained a suitable linear ordering on $\{B_1, \ldots, B_{k+1}\}$, and this induces a geometric permutation on $\{B_1, \ldots, B_{k+1}\}$, too. □

1.4 Existence of common transversals

Finally, we turn to the proof of the main theorem.

Proof of Theorem 1.1. Suppose that $\mathcal{F}_d$ has the property $T(d^2)$, where $d \geqslant 2$. Because of Lemma 1.5, there exists an ordering $\prec$ on $\mathcal{F}_d$ such that for every d^2-element subset $\mathcal{G} \subseteq \mathcal{F}_d$, there is a transversal to $\mathcal{G}$ that intersects the unit balls in the order compatible with $\prec$. Label the elements of $\mathcal{F}_d$ such that $B_1 \prec B_2 \prec \cdots \prec B_m$. Consider the family $\mathcal{C}$ of all the sets $\mathcal{K}(B_{i_1}, \ldots, B_{i_d})$ with $1 \leqslant i_1 < \cdots < i_d \leqslant m$. Theorem 1.2 states that the elements of $\mathcal{C}$ are convex sets of $\mathcal{S}^{d-1}$, and we know from the proof that they are also strongly convex. Lemma 1.5 yields that the intersection of any d elements of $\mathcal{C}$ is not empty.

We show that $\bigcup \mathcal{C} \neq \mathcal{S}^{d-1}$. On the contrary, suppose that x and $(-x)$ are antipodal points of $\mathcal{S}^{d-1}$ covered by $\bigcup \mathcal{C}$, say, $x \in \mathcal{K}(B_{i_1}, \ldots, B_{i_d})$ and $(-x) \in \mathcal{K}(B_{j_1}, \ldots, B_{j_d})$. Then there exists a transversal l to the family

$$\{B_{i_1}, \ldots, B_{i_d}, B_{j_1}, \ldots, B_{j_d}\},$$

and its direction α is in the intersection of the two cones. At the end of the proof of Lemma 1.4 we showed that the angle between x and α and the angle between $(-x)$ and α are less than 1.144, hence the angle between x and $(-x)$ is smaller than $2 \cdot 1.144 < \pi$, a contradiction.

Using this, we can apply the strong version of the Spherical Helly Theorem, see [Deb70], and obtain that $\bigcap \mathcal{C} \neq \emptyset$. So there is a direction $\alpha \in \mathcal{S}^{d-1}$ such that for any d balls in $\mathcal{F}_d$, there is a transversal parallel to α. Let H be a hyperplane in $\mathbb{R}^d$ orthogonal to α, and consider the orthogonal projection of $\mathcal{F}_d$ onto H. The images of the balls will be $(d-1)$–dimensional balls in H, and from the above argument we know that every d of them are intersecting. Hence we are able to apply again Helly's theorem in H and obtain that the intersection of the images contains a point, say, Q. Then the line through Q of direction α is a transversal to the whole family $\mathcal{F}$. $\square$

We remark that there exists a configuration of 3-dimensional unit balls with mutual distances of the centres at least $2\sqrt{2+\sqrt{2}}$, for which $T(3)$ holds but T does not.

To see this, place the balls centred at the points with the following coordinates: $(0, 0, -(1+\varepsilon))$, $(4, 0, 1+\varepsilon)$, $(8, 1+\varepsilon, 0)$, and $(12, -(1+\varepsilon), 0)$. We show that there exists an ε for which the example works. If $\varepsilon = 0$, then the four balls have a common transversal. Hence there exists a maximal ε_0 such that this property holds. At this point there exists exactly one transversal for the four balls. If for any three of the four balls, this transversal is not the only common one, then by the strict convexity of the transversal direction cone, a slightly greater ε is appropriate for the example. Hence it suffices to show that at ε_0, the common transversal is not contained in any of the planes determined by three ball centres. This is implied by the fact that for any such plane, the fourth ball centre is of distance > 1 from the plane, therefore the transversal cannot be contained in it.

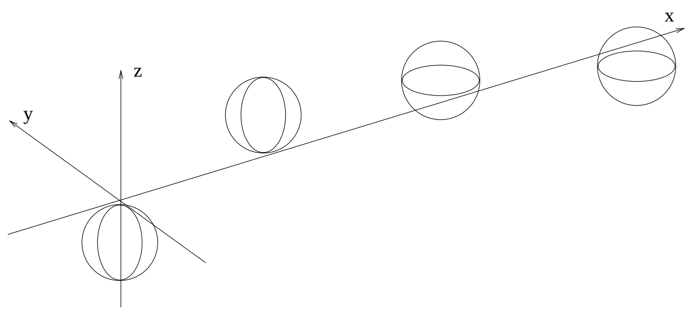

Figure 1.2: Four balls satisfying $T(3)$ with no common transversal

Chapter 2
A lower bound for the Strong Dodecahedral Conjecture

In this chapter, we show that the minimum surface area of a Voronoi cell in a unit ball packing in $\mathbb{R}^3$ is at least $16.1977\ldots$. This result, which is joint with F. Fodor [AF06], provides further support for the Strong Dodecahedral Conjecture according to which the minimum surface area of a Voronoi cell in a 3-dimensional unit ball packing is at least as large as the surface area of a regular dodecahedron of inradius 1, which is about $16.6508\ldots$. In the proof, the cones suspended by the faces of the Voronoi cell are replaced with cones of special types in such a way, that the surface to solid angle ratio does not increase. The obtained configurations belong to a restricted class, in which the minimiser of the surface area is found by standard analytic methods. The minimal configuration has 13 identical faces and one face of a smaller solid angle. However, these faces cannot be joined to form a polytope, which accounts for the error between our estimate and the conjectured extremal value.

2.1 History

One of the most important topics of Discrete Geometry is the theory of packings and coverings. A family $\mathcal{B}$ of unit balls in $\mathbb{R}^3$ forms a *packing* if no two members of $\mathcal{B}$ have a common interior point. We are mostly interested in how dense a packing of unit balls may be, where the *density* of a packing is the proportion of the space covered by the balls. We define this as the limit of the proportion of the volume of the covered part of a ball, where the centre of the ball is fixed and its radius tends to infinity (of course, the limit may not exist). According to a conjecture formulated by Kepler [Kep66], the packing density of unit balls in $\mathbb{R}^3$ is $\pi/\sqrt{18} \approx 0.74078\ldots$, which is attained by a lattice packing. Among lattice packings, this is indeed the best possible, as was shown by Gauss [Gau40].

The quest for proving Kepler's conjecture has been a long saga. Concentrating on relatively new achievements, Rogers [Rog58] showed that the packing density is at most $0.77963\ldots$. This bound was improved by Lindsey [Lin86], and then Muder [Mud88], [Mud93] to 0.773055 .... L. Fejes Tóth sketched a strategy for a proof, which reduces the problem to finitely many cases. Along these lines, the final proof was given by Hales ([Hal05], and 7 other papers).

A central concept in the theory of packing and covering is the *Voronoi cell* of $B \in \mathcal{B}$: this is the set of points $x \in \mathbb{R}^3$ with the property that x is closer to the centre of B than to any other centre in $\mathcal{B}$. It is well known that Voronoi cells are convex polyhedra. Since we are interested in the minimum surface area that a Voronoi cell can have in such a ball packing, we may assume that the packing is reasonably dense, so the Voronoi cell in question is a polytope.

One of the most beautiful problems related to 3-dimensional unit ball packings is the Dodecahedral Conjecture formulated by L. Fejes Tóth [FT43] in 1943. It states that the minimal volume of a Voronoi cell in a 3-dimensional unit ball packing is at least as large as the volume of a regular dodecahedron of inradius 1. This problem has been recently settled in the affirmative by Hales and McLaughlin [HM]. K. Bezdek [Bez00] phrased the following generalised version of the Dodecahedral Conjecture in 2000.

CONJECTURE 2.1 (Strong Dodecahedral Conjecture). *The minimum surface area of a Voronoi cell in a unit ball packing in* $\mathbb{R}^3$ *is at least as large as the surface area of the regular dodecahedron circumscribed about the unit ball, that is* 16.6508 ....

K. Bezdek [Bez00] proved that 16.143 ... is a lower bound for the minimum surface area of the Voronoi cell. To achieve this estimate he used a generalised version of Roger's lemma. Muder [Mud88], [Mud93] developed powerful techniques for estimating the volume of a Voronoi polyhedron in a 3-dimensional sphere packing. K. Bezdek and E. Daróczy-Kiss [BDK05] discovered that Muder's ideas in [Mud88] and [Mud93], after modification, are applicable to the Strong Dodecahedral Conjecture, and thus they improved the lower bound to 16.1445 .... We follow a line of reasoning similar to that of K. Bezdek and E. Daróczy-Kiss [BDK05], combined with ideas formulated by Muder [Mud93].

We shall prove the following statement.

THEOREM 2.2. *The surface area of a Voronoi cell in a unit ball packing in* $\mathbb{R}^3$ *is at least* 16.1977 ....

The geometric idea of the estimate is the following. First, we replace each facecone of the Voronoi cell with a right circular cone or with a shaved right circular cone. During the replacement the solid angle remains the same while the area of the face decreases, and therefore the surface-to-solid angle ratio also decreases. After the replacement, we determine the surface areas of the new cones as a function of the solid angle. Finally, standard analytic tools are applied to find the minimal configuration with respect to this function. The optimal configuration has 13 faces of the same solid angle, and one face with a smaller solid angle. The sum of these solid angles is 4π but the given faces do not form a polytope, hence there is a difference between our lower bound and the conjectured minimal surface area.

2.2 Replacements

First, we define the special cones used in the reduction procedure. A *right circular cone* (RCC) is a cone whose base is a circular disk and its apex lies on the line perpendicular to the disk passing through its center. The radius of the RCC is the radius of its base, while the height is the distance between the apex and the disk. A *shaved circle* is the intersection of a disk and a convex polygon that contains the center of the disk. The *order* of a shaved circle is the number of the segments of its boundary. A shaved circle is vertex-free if there is no vertex of the intersecting polygon in the interior of the circle, and it is regular of order n if the segments of its boundary are n non-intersecting chords of equal length. A *shaved right circular cone* (SRCC) is a cone whose base is a shaved circle and its apex lies on the line perpendicular to the disk and passing through its center. An SRCC is vertex-free or regular if its base has the same property. The *inner radius* of an SRCC is the distance from the center of the disk to the closest side of the shaved circle. The radius of the base circle is the *outer radius* of the SRCC.

The notion of an SRCC is motivated by the fact that a face of a Voronoi polyhedron cannot have vertices and edges 'too close' to its center, moreover, most of the edges must be 'far' from the center: These facts are formulated in the following lemma, whose proof can be found in Muder [Mud93] TL4 and TL5.

LEMMA 2.3 ([Mud93]). *Let $\mathcal{F}$ be a face of a Voronoi polyhedron with center A. Suppose that the distance h between $\mathcal{F}$ and A is at most $\sqrt{2}$. Then no vertex of $\mathcal{F}$ is closer than $\sqrt{3/2}$ to A, no edge of $\mathcal{F}$ is closer than $2/\sqrt{4-h^2}$ to A and at most five edges of $\mathcal{F}$ are less than $\sqrt{3/2}$ from A.*

We say that a facecone of a Voronoi cell is *replaceable* with a second cone with the same apex if the solid angles of the two cones are the same and the area of the base of the second cone is not greater than that of the first one.

In our argument, the facecones are replaced by four types of SRCCs depending on the solid angle. The following important angles separate the different possibilities:

$$0 < \hat{\psi}_1 < \hat{\psi}_2 < \hat{\psi}_3 < 2\pi,$$

with

$$\hat{\psi}_1 = \frac{2}{3}(3 - \sqrt{8}) = 0.3593\ldots,$$

$$\hat{\psi}_2 = 10\sqrt{\frac{2}{3}} \arctan\left(\frac{1}{\sqrt{2}}\right) = 0.9423\ldots,$$

$$\hat{\psi}_3 = \frac{2}{\sqrt{3}}(\sqrt{3} - \sqrt{2})\pi = 1.1529\ldots.$$

Next, we describe the several opportunities used for the reduction steps:

PROPOSITION 2.4. *Any facecone of solid angle ψ is replaceable by an SRCC, having the same solid angle, of the form:*

1. *An RCC of height $h \in [2/\sqrt{3}, \sqrt{2}]$ and radius $(2-h^2)/\sqrt{4-h^2}$ for $\psi \in [0, \hat{\psi}_1]$,*
2. *A regular, vertex-free, order-5 SRCC of height $h \in [1, 2/\sqrt{3}]$, outer radius $\sqrt{\frac{3}{2} - h^2}$, and inner radius $(2-h^2)/\sqrt{4-h^2}$ for $\psi \in [\hat{\psi}_1, \hat{\psi}_2]$,*
3. *A regular, vertex-free, order-5 SRCC of height 1, outer radius $1/\sqrt{2}$, and inner radius $r \in [1/\sqrt{3}, 1/\sqrt{2}]$ for $\psi \in [\hat{\psi}_2, \hat{\psi}_3]$,*
4. *An RCC of height 1 and radius $R \geqslant 1/\sqrt{2}$ for $\psi \in [\hat{\psi}_2, 2\pi]$.*

PROOF. Throughout the proof we will use the following notations. The point A is the center of the Voronoi cell, and for a planar region $\mathcal{R}$ not containing A, $\mathcal{C}(\mathcal{R})$ denotes the cone with base $\mathcal{R}$ and apex A.

First, we state the following:

Let H be a plane not containing A, M the orthogonal projection of A onto H, and P and Q two distinct points in H such that $|MP| > |MQ|$. Let $\mathcal{S}_P$ and $\mathcal{S}_Q$ two *infinitesimal rectangles in H centered at P and Q, respectively, which are translates of each other. Then the solid angle of $\mathcal{S}_P$ is smaller than that of $\mathcal{S}_Q$.* (*)

For the proof of (*), suppose that the distance between A and H is h, and P has distance ϱ from A. A simple computation leads to the fact that the area-to-solid angle ratio of the infinitesimal rectangle centered at P is ϱ^3/h, which is strictly monotonically increasing in ϱ if h is fixed. Since h is the same for P and Q, this implies (*).

Next, we list two basic types of the replacement cones.

Basic replacement schemes. *Let $\mathcal{F}$ be a face of the Voronoi polyhedron with center A, H the plane of $\mathcal{F}$, and M the orthogonal projection of A onto H.*

(R1) Consider a disk $\mathcal{D}$ in H centered at M such that the solid angle of $\mathcal{D}$ is not smaller than the solid angle of $\mathcal{F}$. Let $\mathcal{G}$ be a region of $\mathcal{D}$ containing $\mathcal{D} \cap \mathcal{F}$ and having the same solid angle as $\mathcal{F}$. Then $\mathcal{C}(\mathcal{F})$ is replaceable with $\mathcal{C}(\mathcal{G})$.

(R2) Consider a plane H' parallel to H such that H' separates H and A. Then $\mathcal{C}(\mathcal{F})$ is replaceable with the cone whose base is $H' \cap \mathcal{C}(\mathcal{F})$.

The proof of (R1) is straightforward from (*), while (R2) follows from the fact that the area of $H' \cap \mathcal{C}(\mathcal{F})$ is smaller than the area of $\mathcal{F}$.

In the rest of the proof, we will simply refer to the above schemes as replacements (R1) and (R2).

We divide the proof into three steps. First, if a facecone has height at least $2/\sqrt{3}$, then we replace it with an RCC. For facecones with a smaller height we use either an RCC or a regular, vertex-free, order-5 SRCC with a specified outer radius. In the third step we replace those SRCCs by more specific ones.

Step 1. *Any facecone $\mathcal{C}(\mathcal{F})$ of height $h_0 \geqslant 2/\sqrt{3}$ is replaceable by either*

1. *an RCC of height $h \in [2/\sqrt{3}, \min(h_0, \sqrt{2})]$ and radius $(2-h^2)/\sqrt{4-h^2}$, or*
2. *an RCC of height $2/\sqrt{3}$ and radius $R > 1/\sqrt{6}$.*

Proof of Step 1. Let $\mathcal{S}$ be a disk in the plane of $\mathcal{F}$ centered at M having the same solid angle as $\mathcal{F}$. Let R denote the radius of $\mathcal{S}$. Notice that Lemma 2.3 yields that if $h_0 \leqslant \sqrt{2}$ then $R > (2-h^2)/\sqrt{4-h^2} = \rho(h)$.

Using replacement (R1), $\mathcal{C}(\mathcal{F})$ is replaceable by $\mathcal{C}(\mathcal{S})$. Moreover, (R2) yields that for any $h \in [2/\sqrt{3}, h]$, $\mathcal{C}(\mathcal{S})$ is replaceable by the RCC of height h and radius $R_h = Rh/h_0$.

Note that the function R_h decreases as h decreases. On the other hand, $\rho(h)$ increases as h decreases. Therefore there either exists an $h \in [2/\sqrt{3}, \min(h_0, \sqrt{2})]$ for which $R_h = \rho(h)$, or $R_h > \rho(2/\sqrt{3}) = 1/\sqrt{6}$. These possibilities provide cases (1) and (2) of the statement. $\square$

Step 2. *Any facecone $\mathcal{C}(\mathcal{F})$ of height $h \in [1, 2/\sqrt{3}]$ is replaceable by either*

1. *an RCC of height h and radius $R \geqslant \sqrt{3/2 - h^2}$, or*
2. *a regular, vertex-free, order-5 SRCC of height h, outer radius $\sqrt{3/2 - h^2}$ and inner radius $r \geqslant (2 - h^2)/\sqrt{4 - h^2}$.*

Proof of Step 2. Consider a disk $\mathcal{D}$ of radius $\sqrt{3/2 - h^2}$ centered at M in the plane of $\mathcal{F}$. If the solid angle of $\mathcal{F}$ is greater than that of $\mathcal{D}$, then we can replace $\mathcal{C}(\mathcal{F})$ with an RCC of the same solid angle by replacement (R1), thus obtaining case (1) of Step 2.

Otherwise, Lemma 2.3 yields that $\mathcal{D} \cap \mathcal{F}$ is a vertex-free shaved circle with at most 5 edges. Replacement (R1) allows us to replace $\mathcal{C}(\mathcal{F})$ with an order-5 vertex-free SRCC of outer radius $\sqrt{3/2 - h^2}$ by moving the chords of $\mathcal{D} \cap \mathcal{F}$ away from M.

The only thing that remains to be shown is that this SRCC can be replaced by a regular one of the same solid angle.

For this, consider a circular disk in the plane H centered at M and two infinitesimally narrow parallel chordal bands (regions of the circle bounded by parallel chords) of the same area. Suppose that for the two bands it also holds that they can be rotated around M such that M does not separate them, there bounding chords are parallel, and they are disjoint. Divide both bands into n small pieces, each of them of the same area, by segments perpendicular to the bounding chords, and enumerate these pieces from one end of the bands to the other end. An easy computation shows that the center of the kth such piece is closer to the center of the disk for the chordal band which is closer to M. Moreover, if n is sufficiently large, then the same inequality holds for any two points of the kth pieces. Hence, integrating with the help of (*) yields that the solid angle of the chordal band further from M is smaller. Therefore, if we move infinitesimal chordal bands of the base of the SRCC closer to its center such that the solid angle remains unchanged, than the area of the base decreases. In this way we can transform the SRCC into a regular one, while we do not increase the base area-to-solid angle ratio. Moreover, this operation does not decrease the inner radius of the SRCC. This finishes the proof of Step 2.

□

STEP 3. *Any regular, vertex-free, order-5 SRCC of height* $h_0 \in [1, 2/\sqrt{3}]$*, outer radius* $R \geqslant \sqrt{3/2 - {h_0}^2}$*, and inner radius* $r \geqslant (2 - {h_0}^2)/\sqrt{4 - {h_0}^2}$ *is replaceable by either*

1. *a regular, vertex-free, order-5 SRCC of height* $h \in (1, h_0]$*, outer radius* $\sqrt{3/2 - h^2}$*, and inner radius* $(2 - h^2)/\sqrt{4 - h^2}$*, or*
2. *a regular, vertex-free, order-5 SRCC of height* 1*, outer radius* $1/\sqrt{2}$*, and inner radius* $r \in (1/\sqrt{3}, 1/\sqrt{2}]$*, or*
3. *an RCC of height 1 and radius* $R \geqslant 1/\sqrt{2}$*.*

PROOF OF STEP 3. Denote the SRCC by $\mathcal{C}(\mathcal{S})$. Let R be the radius such that the RCC of height h_0 and radius R has the same solid angle as $\mathcal{C}(\mathcal{S})$. If $R \leqslant \sqrt{3/2 - {h_0}^2}$ then we can replace $\mathcal{C}(\mathcal{S})$ with an SRCC having the same properties as $\mathcal{S}$, and with outer radius $\sqrt{3/2 - {h_0}^2}$ using a (R1), just as in Step 2. Otherwise let

$$h = \max\left(1, h_0\sqrt{\frac{3}{2({h_0}^2 + R^2)}}\right).$$

Notice that $h < h_0$. We use replacement (R2) for the plane at distance h from A to obtain either an RCC of height 1 and radius greater than $1/\sqrt{2}$ (case (3)), or an RCC of outer radius $\sqrt{3/2 - h^2}$.

Therefore, it suffices to consider only the SRCCs with outer radius $\sqrt{3/2 - {h_0}^2}$ and height $h_0 > 1$. We may also suppose that the inner radius $r > (2 - {h_0}^2)/\sqrt{4 - {h_0}^2}$ because otherwise we have the statement of case (1).

Hence, to obtain the statements of Step 3 the inner radius must be decreased. Let $\mathcal{S}$ denote the base of the SRCC, which is a regular shaved circle of order 5. We apply a transformation on $\mathcal{C}(\mathcal{S})$ in the following way: Let $\mathcal{G}$ denote the ball of radius $\sqrt{3/2}$ centered at A, and H_h the plane parallel to the plane of $\mathcal{S}$ and at a distance h from A. Let $\mathcal{S}_h$ be the shaved circle homothetic to $\mathcal{S}$ inscribed in $\mathcal{G} \cap H_h$, and consider the SRCC that is based on $\mathcal{S}_h$. Clearly, for any $h \in [1, h_0)$ the solid angle of $\mathcal{S}_h$ is greater than that of $\mathcal{S}$. Moreover, we will show that the base area-to solid angle ratio of $\mathcal{C}(\mathcal{S}_h)$ is smaller than that of $\mathcal{C}(\mathcal{S})$. Then we may decrease the inner radius of $\mathcal{S}_h$ to obtain an SRCC with the same solid angle as $\mathcal{S}$. There are two possibilities: either there exists an h for which the decreased inner radius is $(2-h^2)/\sqrt{4-h^2}$, when we obtain the statement of case (1), or the decreased inner radius at $h=1$ is at least $1/\sqrt{3}$, which gives the statement of case (2).

Therefore, we only have to show that for any $h \in [1, h_0)$ the base area-to-solid angle ratio of $\mathcal{C}(\mathcal{S}_h)$ is smaller than that of $\mathcal{C}(\mathcal{S})$. Denote by $\mathcal{S}^*$ the radial projection of $\mathcal{S}$ onto bd $\mathcal{G}$. Consider the homothety τ taking $\mathcal{S}$ into $\mathcal{S}_h$. It suffices to show that the surface area of $\mathcal{S}_h^*$ is greater than the surface area of $\tau(\mathcal{S}^*)$. It is easy to check that the image of $\tau(\mathcal{S}^*)$, under the projection onto $\mathcal{G}$ perpendicular to H_h, is contained in $\mathcal{S}_h^*$. Moreover, along this projection at each point of $\tau(\mathcal{S}^*)$, the slope of the tangent plane increases, therefore the surface area of the infinitesimal surface element also increases. Altogether, we obtain that the surface area of $\tau(\mathcal{S}^*)$ is smaller than that of $\mathcal{S}_h^*$. □

Now, we assemble Steps 1-3 and prove Proposition 2.4. If the height $h_0 > 2/\sqrt{3}$ then we apply Step 1, yielding either possibility (1), which is the same as case (1) of Proposition 2.4, or possibility (2), when we can apply Step 3 (note that an RCC is also a regular, vertex-free SRCC). Notice that the three cases of Step 3 are the same as cases (2)-(4) of Proposition 2.4.

If $h_0 \leqslant 2/\sqrt{3}$ then we apply Step 2 and then Step 3 as before, concluding the last three cases of Proposition 2.4. □

2.3 Approximation

First, we determine the base areas and the solid angles of the regular SRCC's obtained in Steps 1–3. The following lemma is the result of elementary computation.

Lemma 2.5. *Let $h, r > 0$, $r \leqslant R$, and let n be a positive integer such that*

$$n \arctan \sqrt{\frac{R^2 - r^2}{r^2}} \leqslant \pi.$$

Then the vertex-free regular SRCC of order n, height h, inner radius r, and outer radius R has base area

$$\sigma_n(h, r, R) = nr\sqrt{R^2 - r^2} + R^2 \left(\pi - n \arctan \sqrt{\frac{R^2 - r^2}{r^2}} \right),$$

and solid angle

$$\psi_n(h,r,R) = 2\left(\pi - n\arctan\left(\sqrt{\frac{h^2}{R^2+h^2}}\sqrt{\frac{R^2-r^2}{r^2}}\right)\right) - 2\sqrt{\frac{h^2}{R^2+h^2}}\left(\pi - n\arctan\sqrt{\frac{R^2-r^2}{r^2}}\right).$$

Thus, by Proposition 2.4, the following holds.

PROPOSITION 2.6. *A facecone of solid angle ψ has base surface area at least $\sigma(\psi)$ where $\sigma(\psi)$ is defined by the following parametric equations:*

$$\sigma_h = \pi\frac{(2-h^2)^2}{4-h^2}, \qquad \psi_h = \pi\left(2 - h\sqrt{4-h^2}\right)$$

for $h \in [2/\sqrt{3}, \sqrt{2}]$, in which case ψ_h parametrises all values in $[0, \hat{\psi}_1]$;

$$\sigma_h = \frac{5(2-h^2)\sqrt{4-3h^2}}{\sqrt{2}(4-h^2)} + \frac{3-2h^2}{2}\left(\pi - 5\arctan\frac{\sqrt{4-3h^2}}{\sqrt{2}(2-h^2)}\right),$$
$$\psi_h = 2\left(\pi - 5\arctan\frac{h\sqrt{4-3h^2}}{\sqrt{3}(2-h^2)}\right) - 2\sqrt{\frac{2}{3}}h\left(\pi - 5\arctan\frac{\sqrt{4-3h^2}}{\sqrt{2}(2-h^2)}\right)$$

for $h \in [1, 2/\sqrt{3}]$, in which case ψ_h parametrises all values in $[\hat{\psi}_1, \hat{\psi}_2]$;

$$\sigma_r = 5r\frac{\sqrt{1-2r^2}}{\sqrt{2}} + \frac{1}{2}\left(\pi - 5\arctan\sqrt{\frac{1-2r^2}{2r^2}}\right),$$
$$\psi_r = 2\left(\pi - 5\arctan\sqrt{\frac{1-2r^2}{3r^2}}\right) - 2\sqrt{\frac{2}{3}}\left(\pi - 5\arctan\sqrt{\frac{1-2r^2}{2r^2}}\right)$$

for $r \in [1/\sqrt{3}, 1/\sqrt{2}]$, in which case ψ_r parametrises all values in $[\hat{\psi}_2, \hat{\psi}_3]$;

$$\sigma_R = R^2\pi, \qquad \psi_R = 2\pi\left(1 - \frac{1}{\sqrt{R^2+1}}\right).$$

for $R \geqslant 1/\sqrt{2}$, in which case ψ_R parametrises all values in $[\hat{\psi}_3, 2\pi]$.

In the subsequent part of the article we shall use the following lemma repeatedly.

Lemma 2.7 ([Mud93]). *Suppose that the function $\sigma(\psi)$ is parametrised by t in the interval (a,b). Then $\sigma(\psi)$ is convex on (a,b) if*

$$\frac{d^2\sigma}{dt^2}\left(\frac{d\psi}{dt}\right)^2 \geqslant \frac{d\sigma}{dt}\frac{d^2\psi}{dt^2}\frac{d\psi}{dt},$$

and concave if

$$\frac{d^2\sigma}{dt^2}\left(\frac{d\psi}{dt}\right)^2 \leqslant \frac{d\sigma}{dt}\frac{d^2\psi}{dt^2}\frac{d\psi}{dt}$$

for all t such that $\psi(t) \in (a,b)$.

In what follows, we simplify the function $\sigma(\psi)$ and determine a minimal configuration with respect to this substitute function.

Proposition 2.8. *The function $\sigma(\psi)$ can be approximated from below by*

$$B(\psi) = \begin{cases} \frac{\sigma(\hat{\psi}_1)}{\hat{\psi}_1}\psi, & \text{if } \psi \in [0,\hat{\psi}_1] \\ \sigma(\hat{\psi}_1) + \frac{\sigma(\hat{\psi}_2)-\sigma(\hat{\psi}_1)}{\hat{\psi}_2-\hat{\psi}_1}(\psi-\hat{\psi}_1), & \text{if } \psi \in [\hat{\psi}_1,\hat{\psi}_2] \\ \sigma(\psi), & \text{if } \psi \in [\hat{\psi}_2,\hat{\psi}_3] \\ \sigma(\psi_3) + \sigma'_+(\hat{\psi}_3)(\psi-\hat{\psi}_3), & \text{if } \psi \in [\hat{\psi}_3, 2\pi] \end{cases}$$

Proof. Since $B(\psi)$ is linear on all intervals except for $[\hat{\psi}_2,\hat{\psi}_3]$, it suffices to show that $\sigma(\psi)$ is concave on $(0,\hat{\psi}_1)$ and $(\hat{\psi}_1,\hat{\psi}_2)$, and convex on $(\hat{\psi}_3, 2\pi)$. We are going to look at these intervals one by one and use the statements of Lemma 2.7 to check the convexity of $B(\psi)$.

In the following part of the proof we simply state the derivatives of σ and ψ with respect to h.

First, let $\psi \in [0,\hat{\psi}_1]$. Then

$$\begin{aligned} \frac{d\sigma}{dh} &= \frac{-2\pi(2-h^2)h(6-h^2)}{(4-h^2)^2}, \\ \frac{d^2\sigma}{dh^2} &= \frac{2\pi(-48+60h^2-12h^4+h^6)}{(4-h^2)^3}, \\ \frac{d\psi}{dh} &= \frac{-2\pi(2-h^2)}{\sqrt{4-h^2}}, \\ \frac{d^2\psi}{dh^2} &= \frac{2\pi h(6-h^2)}{(4-h^2)^{3/2}}. \end{aligned}$$

From the above formulas it follows by simple computation that

$$\frac{d^2\sigma}{dh^2}\frac{d\psi}{dh} - \frac{d\sigma}{dh}\frac{d^2\psi}{dh^2} = \frac{96(2-h^2)^2\pi^2}{(4-h^2)^2(4-h^2)^{3/2}}. \tag{2.1}$$

The right hand side of (2.1) is positive, while $d\psi/dh$ is negative on the whole interval $[0, \hat{\psi}_1]$, which implies that $\sigma(\psi)$ is concave on the designated interval.

Next, let $\psi \in [\hat{\psi}_1, \hat{\psi}_2]$. Introduce the following notation.

$$\gamma(h) := \pi - 5\arctan\left(\frac{\sqrt{4-3h^2}}{\sqrt{2}(2-h^2)}\right)$$

Note that $\gamma(h) > 0$ on the designated interval. Furthermore,

$$\begin{aligned}
\frac{d\psi}{dh} &= \frac{-2\sqrt{2}}{\sqrt{3}(4-h^2)}\left(5\sqrt{2}\sqrt{4-3h^2} + (4-h^2)\gamma(h)\right), \\
\frac{d^2\psi}{dh^2} &= \frac{20h(20-13h^2-3h^4)}{\sqrt{3}\sqrt{4-3h^2}(3-2h^2)(4-h^2)^2}, \\
\frac{d\sigma}{dh} &= \frac{-h}{(4-h^2)^2}\left(5\sqrt{2}\sqrt{4-3h^2}(6-h^2) + 2(4-h^2)^2\gamma(h)\right), \\
\frac{d^2\sigma}{dh^2} &= \frac{-10\sqrt{2}(144-308h^2+196h^4-34h^6+3h^8)}{(4-h^2)^3(3-2h^2)\sqrt{4-3h^2}} - 2\gamma(h).
\end{aligned}$$

We need to prove that the following expression is positive:

$$\begin{aligned}
&\frac{d^2\sigma}{dh^2}\frac{d\psi}{dh} - \frac{d\sigma}{dh}\frac{d^2\psi}{dh^2} = \\
&\qquad = \frac{4}{\sqrt{3}(4-h^2)^4(4-3h^2)(3-2h^2)}\left(\xi_0(h) + \xi_1(h)\gamma(h) + \xi_2(h)\gamma^2(h)\right),
\end{aligned}$$

where

$$\begin{aligned}
\xi_0(h) &= 25\sqrt{2}(4-3h^2)(288-496h^2+294h^4-73h^6+9h^8), \\
\xi_1(h) &= 20\sqrt{4-3h^2}(4-h^2)(3-2h^2)(56-62h^2+20h^4-3h^6), \\
\xi_2(h) &= \sqrt{2}(4-h^2)^4(3-2h^2)(4-3h^2).
\end{aligned}$$

Introducing $w = 4-h^2$, $x = 4-3h^2$, $y = h^2-1$, $z = 3-2h^2$, we obtain

$$\begin{aligned}
\xi_2(h) &= \sqrt{2}w^4 z \geqslant 0,\\
\xi_1(h) &= 20\sqrt{x}wz(2z + 9x + 10y^2 + xy^2) \geqslant 0,\\
\xi_0(h) &= 25\sqrt{2}x(4 + y + 2xw + 12x^2 + yw^2 + 12y^2x + 9y^4) \geqslant 0.
\end{aligned}$$

Since all of these terms are positive, and $d\psi/dh$ is negative on the whole interval $[\hat{\psi}_1, \hat{\psi}_2]$, we obtain that $\sigma(\psi)$ is concave.

Finally, let $\psi \in [\hat{\psi}_3, 2\pi]$. An easy computation yields that in the considered interval

$$\sigma'(\psi) = \left(\frac{d\sigma}{d\psi}\right)_{\psi=\psi_R} = (R^2 + 1)^{3/2},$$

which is a monotonically increasing function, therefore $\sigma(\psi)$ is convex on $(\hat{\psi}_3, 2\pi)$. Also notice that the right-hand derivative of $\sigma(\psi)$ at $\hat{\psi}_3$ is $\sqrt{27/8}$. □

2.4 The minimal configuration

In this final section, we determine the minimal configuration with respect to the function $B(\psi)$, leading to the lower bound formulated in Theorem 2.2.

PROPOSITION 2.9. *Suppose $\psi_i \in [0, 2\pi]$ for every $i \in [n]$, and $\sum_{i=0}^{n} \psi_i = 4\pi$. Moreover, let $\psi_m = 4\pi - 13\hat{\psi}_2$. Then*

$$\sum_{i=0}^{n} B(\psi_i) \geqslant B(\psi_m) + 13B(\hat{\psi}_2).$$

PROOF. First, we prove that $B(\psi)$ is concave on $(0, \hat{\psi}_2)$. To see this, it suffices to compute that the slope of $B(\psi)$ on $(0, \hat{\psi}_1)$ is

$$\frac{\sigma(\hat{\psi}_1)}{\hat{\psi}_1} = \frac{1}{12 - 8\sqrt{2}} = 1.4571\ldots,$$

and on $(\hat{\psi}_1, \hat{\psi}_2)$ it is

$$\frac{\sigma(\hat{\psi}_2) - \sigma(\hat{\psi}_1)}{\hat{\psi}_2 - \hat{\psi}_1} = \frac{5\sqrt{2} + 2\pi - 15\arctan(1/\sqrt{2})}{-2(5\pi + 2\sqrt{6}\pi) - 4\sqrt{2}\pi - 10\sqrt{6}\arctan(1/\sqrt{2}))} = 1.17834\ldots,$$

which implies concaveness.

Now we show that $B(\psi)$ is convex on $(\hat{\psi}_2, 2\pi)$. It suffices to show that $\sigma(\psi)$ is convex on $(\hat{\psi}_2, \hat{\psi}_3)$ and that the right- and left-hand derivatives of $\sigma(\psi)$ at $\hat{\psi}_3$ are equal.

A straightforward computation, based on Proposition 2.6, yields that the derivative of $\sigma(\psi)$ on $(\hat{\psi}_2, \hat{\psi}_3)$ is

$$\sigma'(\psi) = \left(\frac{d\sigma}{d\psi}\right)_{\psi=\psi_r} = (1+r^2)\frac{\sqrt{3}}{\sqrt{2}}.$$

Thus, $\sigma(\psi)$ is indeed convex on $(\hat{\psi}_2, \hat{\psi}_3)$, and its the left-hand derivative at $\hat{\psi}_3$ is $3\sqrt{3/8} = \sqrt{27/8}$, the same as the right-hand derivative.

We remark that there is an alternative way to compute $\sigma'(\psi)$ on $(\hat{\psi}_2, \hat{\psi}_3)$. The cones that belong to this interval are regular, vertex-free, order-5 SRCC's of height 1, outer radius $1/\sqrt{2}$, and inner radius $r \in [1/\sqrt{3}, 1/\sqrt{2}]$. As we increase r, the new part of the SRCC is the union of five cones over chordal bands of the base circle, and the derivative is the base area-to-solid angle ratio of these cones. This computation leads to the same result as above.

Hence, $B(\psi)$ is concave on $(0, \hat{\psi}_2)$ and convex on $(\hat{\psi}_2, 2\pi)$. Therefore, if we are given two angles $0 < \psi_0 < \psi_1 < \hat{\psi}_2$, then

$$B(\psi_0) + B(\psi_1) \geqslant B(\psi_0 - t) + B(\psi_1 + t),$$

where $t = \min(\psi_0, \hat{\psi}_2 - \psi_1)$. Therefore, in the sum $\sum_{i=0}^{n} B(\psi_i)$, we can eliminate the angles smaller than $\hat{\psi}_2$ except for at most one of them. Moreover, the convexity of $B(\psi)$ yields that the angles that are at least $\hat{\psi}_2$ can be replaced by identical angles, and since $14\hat{\psi}_2 > 4\pi$, we also know that there are at most 13 of these. Altogether, we can replace any configuration of angles by another configuration consisting of at most one angle smaller than $\hat{\psi}_2$ and at most 13 identical angles which are at least $\hat{\psi}_2$. In summary, we obtain the following statement.

Let $k \leqslant 13$ and

$$\beta_k(\psi) = B(\psi) + kB\left(\frac{4\pi - \psi}{k}\right)$$

for all $\psi \in [0, \min(\hat{\psi}_2, 4\pi - h\hat{\psi}_2)$. Then

$$\sum_{i=0}^{n} B(\psi_i) \geqslant \beta_k(\psi)$$

for some $k \leqslant \min(n, 13)$ *and* $\psi \in [0, \min(\hat{\psi}_2, 4\pi - k\hat{\psi}_2))$. (†)

In what follows, we will find the minimal configuration with respect to β. The first step in this process is to prove that it is sufficient to consider $\beta_{13}(\psi)$.

If $k < 13$*, then* $\beta_k(\psi) \geqslant \beta_{k+1}(0)$. (‡)

For the proof of (‡), we introduce the following function.

$$g(\psi) = \begin{cases} \frac{\sigma(\hat{\psi}_2)}{\hat{\psi}_2}\psi, & \text{if } \psi \in [0, \hat{\psi}_2) \\ B(\psi), & \text{if } \psi \in [\hat{\psi}_2, 2\pi]. \end{cases}$$

The concavity of $B(\psi)$ on $[0, \hat{\psi}_2]$ implies that $g(\psi) \leqslant B(\psi)$ everywhere in $[0, 2\pi]$. Furthermore, by definition, the right-hand derivative of g at $\hat{\psi}_2$ is

$$\sigma'_+(\hat{\psi}_2) = \frac{4}{3}\frac{\sqrt{3}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{3}} = 1.6329\ldots.$$

Since on the interval $(0, \hat{\psi}_2)$

$$g'(\psi) = \frac{\sigma(\hat{\psi}_2)}{\hat{\psi}_2} = \frac{5\sqrt{2} + 3\pi - 15\arctan(1/\sqrt{2})}{(2 - 4\sqrt{6}\pi + 20\sqrt{6}\arctan(1/\sqrt{2}))} = 1.2846\ldots,$$

we may conclude that $g(\psi)$ is convex on $[0, 2\pi)$. Hence

$$\beta_k(\psi) \geqslant g(\psi) + k\, g\left(\frac{4\pi - \psi}{k}\right) \geqslant (k+1)\, g\left(\frac{4\pi}{k+1}\right).$$

Moreover, $k < 13$ implies that $4\pi/(k+1) > \hat{\psi}_2$ and so

$$(k+1)\, g\left(\frac{4\pi}{k+1}\right) = (k+1)\, B\left(\frac{4\pi}{k+1}\right) = \beta_{k+1}(0),$$

which proves (‡).

Thus, we obtained that if $k < 13$ then $\beta_k(\psi) \geqslant \beta_{13}(0)$, and so, by (†), a lower bound for $\beta_{13}(\psi)$ is also a lower bound for $\sum_{i=0}^{n} B(\psi_i)$.

Now, we shall look for the minimum of $\beta_{13}(\psi)$.

$$\beta'_{13}(\psi) = B'(\psi) - \sigma'(\frac{4\pi - \psi}{13})$$

for $\psi \in [0, 4\pi - 13\hat{\psi}_2] \subset [0, \hat{\psi}_1]$. On this interval, $B(\psi)$ is linear, and

$$B'(\psi) = \frac{\sigma(\hat{\psi}_1)}{\hat{\psi}_1} = 1.4571\ldots.$$

On the other hand,

$$\frac{4\pi - \psi}{13} \in [\hat{\psi}_2, \frac{4\pi}{13}] \subset [\hat{\psi}_2, \hat{\psi}_3],$$

so $\sigma'((4\pi - \psi)/13) = (1 + r^2)\sqrt{3/2}$. The angle ψ varies between $\hat{\psi}_2$ and $4\pi/13$, so the inner radius r is at least $1/\sqrt{3}$, and

$$\sigma'\left(\frac{4\pi - \psi}{13}\right) = (1 + r^2)\sqrt{3/2} \geqslant 2\sqrt{2}/\sqrt{3} = 1.6329\ldots.$$

Therefore, $\beta'_{13}(\psi)$ is negative in the entire interval. It follows that $\beta_{13}(\psi)$ is minimal when ψ is maximal, that is, when $\psi = \psi_m = 4\pi - 13\hat{\psi}_2$.

□

Proof of Theorem 2.2. Using the exact value of $\hat{\psi}_2$, we obtain that

$$\psi_m = -\frac{\pi}{3} + \frac{26}{3}\sqrt{6}(\pi - 5\arctan(1/\sqrt{2})) = 0.3155\ldots.$$

After substituting the value of $\hat{\psi}_2$ and ψ_m, the new bound for the minimum surface area of the Voronoi cell is

$$\beta(\psi_m) + 13\beta(\hat{\psi}_2) = \frac{\sigma(\hat{\psi}_1)}{\hat{\psi}_1}\psi_m + 13\sigma(\hat{\psi}_2) = 16.1977\ldots.$$

□

Chapter 3

Stability results for the volume of random simplices

It is known that for a convex body K in $\mathbb{R}^d$ of volume one, the expected volume of random simplices in K is minimised if K is an ellipsoid, and for $d = 2$, maximised if K is a triangle. In this chapter, we provide corresponding stability estimates in terms of the Banach-Mazur distance of K from the ellipsoid and the triangle. In Section 3.1, the long history and various connections of the problem are presented. The main results are listed in Section 3.2. The core technical lemmas leading to these are proved in Section 3.3. The next section contains the proof of the stability of the lower bound, which is obtained first for centrally symmetric, then for general convex bodies. The argument estimates the change of the expectation when applying one step of Steiner symmetrisation. The stability of the upper bound in the plane is shown in Section 3.5, using linear shadow systems. We conclude the chapter by showing that our results also imply the stability of the Petty projection inequality. These results are a joint work with K. J. Böröczky [AB].

3.1 The saga

Let K be a convex body in $\mathbb{R}^d$. What is the expected value of the volume of a random simplex in K? Naturally, this question needs to be clarified further. We will work with two (or three) models: in the first, all the vertices of the simplex are chosen uniformly and independently from K, while in the second, one vertex is at a fixed position – in a special case, this is the centroid of K. We are interested in other moments as well, and also, we would like the answer to be invariant under affine transformations.

As a general reference for stochastic geometry, we refer to R. Schneider, W. Weil [SW08], and for convexity, to T. Bonnesen, W. Fenchel [BF87], P.M. Gruber [Gru07] and R. Schneider [Sch93]. V or V_d stands for the d-dimensional volume (if the dimension is clear, we shall omit d), the convex hull of the points $x_1 \ldots, x_n$ is denoted by $[x_1, \ldots, x_n]$, and $\gamma(K)$ is the centroid of K.

Definition 3.1. Let K be a convex body in $\mathbb{R}^d$. For any $n \geqslant d+1$ and $p > 0$, let

$$\mathbb{E}_n^p(K) = V(K)^{-n-p} \int_K \dots \int_K V([x_1, \dots, x_n])^p \, dx_1 \dots dx_n.$$

Further, for a fixed $x \in \mathbb{R}^d$, let

$$\mathbb{E}_x^p(K) = V(K)^{-d-p} \int_K \dots \int_K V([x, x_1, \dots, x_d])^p \, dx_1 \dots dx_d.$$

Specifically, we write $\mathbb{E}_*^p(K)$ for $\mathbb{E}_x^p(K)$, when $x = \gamma(K)$.

In particular, for integer p, $\mathbb{E}_{d+1}^p(K)$ is the expectation of the pth moment of the relative volume of simplices in K. Clearly, $\mathbb{E}_n^p(K)$ and $\mathbb{E}_*^p(K)$ are invariant under non-singular affine transformations, and $\mathbb{E}_o^p(K)$ is invariant under non-singular linear transformations, where o stands for the origin. We note that for fixed K and $p \geqslant 1$, $\mathbb{E}_x^p(K)$ is a strictly convex function of x, therefore it attains its minimum at a unique point. If K is o-symmetric, then the minimum is attained at o, and $\mathbb{E}_o^p(K) = \mathbb{E}_*^p(K)$.

In the rest of the section, we give an overview of the history of the quantities introduced in Definition 3.1 and their various connections. The main results are presented in Section 3.2, whose proofs are found in the subsequent parts. Section 3.6 contains further corollaries.

3.1.1. *Sylvester's problem.* The quantity $\mathbb{E}_{d+1}^p(K)$ arose right at the first steps of random convex geometry. Indeed, (probably) the first question in this topic is due to Sylvester [Syl64]: in 1864 he (vaguely) asked, what is the probability that four randomly chosen points in a planar convex disc are in convex position, that is, none of them is in the convex hull of the other three. Generalising to higher dimensions, if $d+2$ points are chosen randomly from a convex body $K \subset \mathbb{R}^d$, then the sought quantity is exactly $1 - (d+2)\mathbb{E}_{d+1}^1(K)$. It is then natural to ask: for which convex bodies is this probability minimal and maximal? The first steps in this direction were taken by Blaschke ([Bla17] and [Bla23]), who showed that in the plane, the probability in question is maximal for ellipses, and minimal for triangles. The maximisers in higher dimensions are the ellipsoids (cf. Groemer [Gro73]), whereas the minimiser bodies in higher dimensions are still not known. We shall state these results as theorems later. For a thorough historical account of this problem, see Klee [Kle69], and also Bárány [Bár08].

3.1.2. *Minimisers and affine inequalities.* Let $K \subset \mathbb{R}^d$ be a convex body with $\gamma(K) = 0$. The *intersection body* IK of K is defined by its radial function:

$$\rho_{IK}(u) = V_{d-1}(K \cap u^{\perp}).$$

H. Busemann [Bus53] established the formula

$$V_d(K)^{d-1} = \frac{(d-1)!}{2} \int_{S^{d-1}} V_{d-1}(K \cap u^{\perp})^d \, \mathbb{E}_o^1(K \cap u^{\perp}) d\sigma(u), \tag{3.1}$$

where σ is surface area measure on S^{d-1}. In the same paper, he proved the *Busemann random simplex inequality*:

$$\mathbb{E}_o^1(K) \geqslant \mathbb{E}_o^1(B^d). \tag{3.2}$$

Combining (3.1) and (3.2), he derived the *Busemann intersection inequality*, stating that the volume of the intersection body is maximal for ellipsoids:

$$V_d(IK) \leqslant \frac{\kappa_{d-1}^d}{\kappa_d^{d-2}} V(K)^{d-1}, \tag{3.3}$$

where $\kappa_d = V_d(B^d)$.

A couple of years later, Petty [Pet61] introduced *centroid bodies*: the centroid body ΓK of K is the convex body in $\mathbb{R}^d$ defined by the support function

$$h_{\Gamma K}(u) = \frac{1}{V(K)} \int_K |\langle u, x \rangle| dx.$$

Using an approximation argument and the volume formula for zonotopes, he obtained the following formula for the volume of ΓK:

$$V_d(\Gamma K) = 2^d V_d(K) \mathbb{E}_o^1(K). \tag{3.4}$$

The argument is nicely presented in [Gar06]. Using the Busemann random simplex inequality (3.2), Petty obtained the *Busemann-Petty centroid inequality*, which states that the volume of the centroid body is minimal for ellipsoids:

$$\frac{V_d(\Gamma K)}{V_d(K)} \geqslant \left(\frac{2\kappa_{d-1}}{(d+1)\kappa_d} \right)^d. \tag{3.5}$$

The conjectured converse of this inequality is that the volume is maximised for simplices provided that o is the centroid; this would be crucial in high dimensional convex geometry, as we shall soon see.

The minimisers of the mean volumes of random simplices are known in full generality: they are the ellipsoids for all the quantities introduced in Definition 3.1.

THEOREM 3.2 (Blaschke, Busemann, Groemer). *For any convex body K in $\mathbb{R}^d$, for any $p \geqslant 1$, and for any $n \geqslant d+1$, we have*

$$\mathbb{E}_o^p(K) \geqslant \mathbb{E}_o^p(B^d) \quad \textit{and} \quad \mathbb{E}_*^p(K) \geqslant \mathbb{E}_*^p(B^d) \quad \textit{and} \quad \mathbb{E}_n^p(K) \geqslant \mathbb{E}_n^p(B^d).$$

Here $\mathbb{E}_o^p(K) = \mathbb{E}_o^p(B^d)$ if and only if K is an o-symmetric ellipsoid, and $\mathbb{E}_*^p(K) = \mathbb{E}_*^p(B^d)$ or $\mathbb{E}_n^p(K) = \mathbb{E}_n^p(B^d)$ if and only if K is an ellipsoid.

As we noted before, Blaschke [Bla23] handled $\mathbb{E}_3^1(K)$ in the planar case, Groemer [Gro73] extended his result to higher dimensions, H. Busemann [Bus53]) obtained the estimate for $\mathbb{E}_o^p(K)$. Groemer [Gro74] derived the result for $\mathbb{E}_n^p(K)$. All the proofs are similar and based on Steiner symmetrisation. For thorough discussions of these inequalities and relatives, see the survey article [Lut93] by E. Lutwak, or the monograph [Gar06] by R.J. Gardner. The minimal values in the cases of random simplices when $p \geqslant 1$ is an integer, can be found as Theorems 8.2.2 and 8.2.3 in R. Schneider, W. Weil [SW08]. Writing $\kappa_d = V(B^d) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$, we have

$$\begin{aligned}
\mathbb{E}_*^p(B^d) &= \binom{d+p}{d}^{-1} \kappa_d^{-d-p} \kappa_{d+p}^d \cdot \frac{\kappa_1 \dots \kappa_d}{\kappa_{p+1} \dots \kappa_{p+d}} \\
\mathbb{E}_{d+1}^p(B^d) &= (d!)^{-p} \binom{d+p}{d}^{-1} \kappa_d^{-d-p-1} \kappa_{d+p}^{d+1} \cdot \frac{\kappa_{d^2+dp+d}}{\kappa_{d^2+dp+d+p}} \cdot \frac{\kappa_1 \dots \kappa_d}{\kappa_{p+1} \dots \kappa_{p+d}}
\end{aligned}$$

3.1.3. *Maximum inequalities and the slicing conjecture.* As usual, let K be a convex body in $\mathbb{R}^d$, and assume that $\gamma(K) = o$. The *inertia matrix* of K is the $d \times d$ matrix M given by

$$M_{ij} = \int_K x_i x_j \, dx,$$

where x_i is the ith coordinate of x. Since for any $y \in \mathbb{R}^d$, we have $y^\top My = \int_K \langle x, y \rangle^2 dx$, it follows that M is a positive definite, symmetric matrix, and hence it has a positive square-root A. The inertia matrix of the convex body $A^{-1}K$ is then $I_d / \det A$ (see J. Bourgain, M. Meyer, V. Milman, A. Pajor [BMMP88], and for a more detailed discussion, see Ball [Bal88]). For a non-singular affine transformation $\Phi \in GL_d$, we say that ΦK is in *isotropic position* with the constant of isotropy L_K, if $\gamma(K) = o$, $V_d(\Phi K) = 1$, and the inertia matrix of ΦK is a multiple of the identity, that is,

$$\int_{\Phi K} \langle x, y \rangle^2 dx = L_K^2 \|y\|^2$$

for every $y \in \mathbb{R}^d$. We just have seen that every convex body has a non-singular affine image that is in isotropic position, and it is well known that the isotropic position is unique up to orthogonal transformations. Hence, L_K is an affine invariant. Moreover,

$$L_K = (\det M)^{1/2d} V_d(K)^{-(d+2)/2d}. \tag{3.6}$$

By expanding the determinant of M, one obtains (see Blaschke [Bla18] or Giannopoulos [Gia])

$$\det M = d!\, V_d(K)^{d+2} \mathbb{E}_*^2(K),$$

and hence from (3.6),

$$L_K^{2d} = d!\, \mathbb{E}_*^2(K). \tag{3.7}$$

The *slicing conjecture*, initiated by J. Bourgain [Bou86], asserts that there exists a universal constant L, for which $L_K \leqslant L$ for every convex body K, regardless of the dimension. There are various equivalent formulations of this major open problem; for thorough surveys, consult the papers V.D. Milman and A. Pajor [MP89], and A.A. Giannopoulos and V.D. Milman [GM04] for some later results.

By (3.7), one has to determine the maximum of $\mathbb{E}_*^2(K)$. The most general conjecture is the following, where T^d stands for a d-dimensional simplex:

Conjecture 3.3 (Simplex conjecture). *If K is a convex body in $\mathbb{R}^d$, then for any $p \geqslant 1$ and for any $n \geqslant d+1$,*

$$\mathbb{E}_*^p(K) \leqslant \mathbb{E}_*^p(T^d) \text{ and } \mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(T^d),$$

with equality if and only if K is a simplex.

Little is known about Conjecture 3.3. The proposed extremal values are known explicitly only in a few cases. W.J. Reed [Ree74] proved that if $p \geqslant 1$ is an integer, then

$$\mathbb{E}_3^p(T^2) = \frac{12}{(p+1)^3(p+2)^3(p+3)(2p+5)}\left[6(p+1)^2 + (p+2)^2\sum_{i=0}^{p}\binom{p}{i}^{-2}\right].$$

For all n, only the first moments $\mathbb{E}_n^1(T^2)$ and $\mathbb{E}_n^1(T^3)$ are known, see C. Buchta [Buc84] and C. Buchta, M. Reitzner [BR01], respectively. Even explicit values of $\mathbb{E}_{d+1}^1(T^d)$ for $d \geqslant 4$ are missing. It is important that for any $d \geqslant 2$,

$$\mathbb{E}_*^2(T^d) \leqslant \frac{1}{d!}, \tag{3.8}$$

see Giannopoulos [Gia]. Thus, the simplex conjecture for $\mathbb{E}_*^2(K)$ implies the slicing conjecture.

The method of Dalla and Larman [DL91], who considered $\mathbb{E}_n^1(K)$, combined with Theorem 3.9 of Campi, Colesanti, and Gronchi [CCG99] yields Conjecture 3.3 if K is a polytope of at most $d+2$ vertices. Bárány and Buchta [BB93] proved the following asymptotic version of Conjecture 3.3 for $p=1$. If K is not a simplex, there exists a threshold n_K depending on K, such that $\mathbb{E}_n^1(K) < \mathbb{E}_n^1(T^d)$ for $n > n_K$. Conjecture 3.3 for all K and n is verified only in the plane.

Theorem 3.4 (Blaschke,Dalla-Larman,Giannopoulos). *If K is a planar convex body, then for any $n \geqslant 3$ and $p \geqslant 1$, $\mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(T^2)$, with equality if and only if K is a triangle.*

More precisely, it was proved by Blaschke [Bla17] for $n = 3$, and by Dalla and Larman [DL91] for $n \geqslant 4$, that $\mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(T^2)$. In addition, Giannopoulos [Gia92] verified that equality holds only if K itself is a triangle.

We shall see in Section 3.5 (compare (3.29) and Lemma 3.15) that the method of S. Campi, A. Colesanti, P. Gronchi [CCG99], see Theorem 3.9, leads to the planar version of the first statement of Conjecture 3.3.

Theorem 3.5. *If K is a convex disc, then for any $p \geqslant 1$, we have $\mathbb{E}_*^p(K) \leqslant \mathbb{E}_*^p(T^2)$, with equality if and only if K is a triangle.*

For centrally symmetric planar convex discs and $p = 1$, T. Bisztriczky and K. Böröczky Jr. [BB01] proved the analogue of Theorem 3.5 with o-symmetric parallelograms instead of triangles as maximisers. The method readily extends to all $p \geqslant 1$.

3.1.4. *Equivalence.* Finally, we establish connections between the different quantities measuring the mean volumes of random simplices.

For every $p \geqslant 1$ and for any convex body $K \subset \mathbb{R}^d$, we have

$$(\mathbb{E}_*^p(K))^{1/p} \leqslant (\mathbb{E}_{d+1}^p(K))^{1/p} \leqslant (d+1)(\mathbb{E}_*^p(K))^{1/p}. \tag{3.9}$$

For a proof, see Proposition 1.3.1 of Giannopoulos [Gia].

Specifically, for $p = 2$, one obtains

$$(d+1)\mathbb{E}_*^2(K) = \mathbb{E}_{d+1}^2(K). \tag{3.10}$$

The proof goes by assuming that $\gamma(K) = o$ and K is in isotropic position. Given $x_1, \dots, x_{d+1} \in \mathbb{R}^d$,

$$V_d([x_1, \dots, x_{d+1}]) = \frac{1}{d!} \det((x_1, 1), \dots, (x_{d+1}, 1)).$$

Using this formula and proceeding as in Proposition 3.7. of Milman and Pajor [MP89], one obtains (3.10).

Thus, in view of (3.7), to prove the slicing conjecture, it would suffice to estimate $\mathbb{E}_{d+1}^2(K)$.

Next, we show that all the quantities $\mathbb{E}_*^p(K)$ and $\mathbb{E}_{d+1}^p(K)$ are equivalent in the following sense: for any $p, q > 0$, there exist constants $c_{p,q}$ and $C_{p,q}$ depending on p and q only, such that if $\mathbb{E}^p(K)$ stands for either $\mathbb{E}_*^p(K)$ or $\mathbb{E}_{d+1}^p(K)$, then

$$c_{p,q}^d(\mathbb{E}^p(K))^{1/p} \leqslant (\mathbb{E}^q(K))^{1/q} \leqslant C_{p,q}^d(\mathbb{E}^p(K))^{1/p}. \tag{3.11}$$

To this end, using (3.9), it suffices to show that $\mathbb{E}_*^p(K)$ and $\mathbb{E}_*^q(K)$ are equivalent. Hölder's inequality implies that for $0 < p < q$,

$$(\mathbb{E}_*^p(K))^{1/p} \leqslant (\mathbb{E}_*^q(K))^{1/q}. \tag{3.12}$$

To see the estimate in the other direction, we refer to Milman and Pajor [MP89]. Proposition 3.7 therein states that there exists an absolute constant $c > 0$, such that for any convex body $K \subset \mathbb{R}^d$, and for any $0 < p \leqslant 2$,

$$(\mathbb{E}_*^2(K))^{1/2} \leqslant c^d(\mathbb{E}_*^p(K))^{1/p}. \tag{3.13}$$

The key step is using the concentration of volume property of convex bodies (indeed, for log-concave functions), cf. Borell's lemma, which then establishes that for a fixed $v \in \mathbb{R}^d$, all the L_p-norms $(\int_K |\langle x, v\rangle|^p dx)^{1/p}$ are equivalent. Then, one uses the fact that fixing $x_1, \ldots, x_{d-1}$, $V[x_1, \ldots, x_d]$ is a linear function of x_d, and hence,

$$\mathbb{E}^p_{d+1}(K) = \int_K |\langle x_d, v\rangle|^p dx_d$$

for some $v \in \mathbb{R}^d$, provided $V_d(K) = 1$. Equation (3.13) can then be obtained by an inductive argument, provided K is in isotropic position.

When $p > 2$, then we use the following Khinchine type inequality: if $K \subset \mathbb{R}^d$ is a convex body of volume 1, then for any $v \in \mathbb{R}^d$,

$$\left(\int_K |\langle x, v\rangle|^p dx\right)^{1/p} \leqslant cp \int_K |\langle x, v\rangle|\, dx \leqslant cp \left(\int_K |\langle x, v\rangle|^2 dx\right)^{1/2}$$

for some universal constant c (see Proposition 2.1.1. of Giannopoulos [Gia]). Then the argument of Milman and Pajor works, yielding that there exists a constant C, such that

$$\left(\frac{C}{p}\right)^d (\mathbb{E}^p_*(K))^{1/p} \leqslant (\mathbb{E}^2_*(K))^{1/2}.$$

Referring to (3.12) and (3.13), we arrive to (3.11).

We note that in order to prove the slicing conjecture, using formulas (3.7) and (3.11), it would suffice to verify either the first or the second statement (with $n = d+1$) of Conjecture 3.3 for any particular $p \geqslant 1$.

3.2 Results

Our goal is to provide stability versions of Theorems 3.2, 3.4 and 3.5. We shall use the *Banach-Mazur distance* $\delta_{\mathrm{BM}}(K, M)$ of the convex bodies K and M, which is defined by

$$\delta_{\mathrm{BM}}(K, M) = \min\{\lambda \geqslant 1 : K - x \subset \Phi(M - y) \subset \lambda(K - x) \quad \text{for } \Phi \in \mathrm{GL}_d,\ x, y \in \mathbb{R}^d\}.$$

If K and M are o-symmetric, then $x = y = o$ can be assumed. It follows by Fritz John's ellipsoid theorem that $\delta_{\mathrm{BM}}(K, B^d) \leqslant d$ for any d-dimensional convex body K, and $\delta_{\mathrm{BM}}(K, B^d) \leqslant \sqrt{d}$ holds if K is centrally symmetric. Moreover, J. Lagarias and G. Ziegler verified in [LZ91] that $\delta_{\mathrm{BM}}(K, T^d) \leqslant d+2$.

First, the stability version of Theorem 3.2.

THEOREM 3.6. *If K is a convex body in $\mathbb{R}^d$ with $\delta_{\mathrm{BM}}(K, B^d) = 1+\delta$ for $\delta > 0$, then for any $p \geqslant 1$,*

$$\mathbb{E}_*^p(K) \geqslant (1+\gamma^p \delta^{d+3})\mathbb{E}_o^p(B^d)$$
$$\mathbb{E}_{d+1}^p(K) \geqslant (1+\gamma^p \delta^{d+3})\mathbb{E}_{d+1}^p(B^d),$$

where the constant $\gamma > 0$ depends on d only. Moreover, if K is centrally symmetric, then the error terms can be replaced by $\gamma^p \delta^{(d+3)/2}$.

Similar stability estimates preceded our work. Groemer [Gro94] showed that under rather strict regularity conditions on the boundary of K, the above statement holds with an error term of order $\delta^{c\,d^2}$ for some universal constant c. Fleury, Guédon and Paouris [FGP07] proved a stability result for the mean width of L_p-centroid bodies, which in the case $p = 1$, yields a stability estimate for $\mathbb{E}_o^1(K)$ by (3.4). However, the error term obtained this way is again only of order $\delta^{c\,d^2}$ for some universal constant c. We remark that for $p \neq 1$, no such direct connection exists between $\mathbb{E}_o^p(K)$ and the volume of the L_p-centroid body.

Second, the stability version of Theorems 3.4 and 3.5.

THEOREM 3.7. *If K is a planar convex body with $\delta_{\mathrm{BM}}(K, T^2) = 1+\delta$ for some $\delta > 0$, and $p \geqslant 1$, then*

$$\begin{aligned} \mathbb{E}_*^p(K) &\leqslant (1 - c^p\delta^2)\mathbb{E}_*^p(T^2) \\ \mathbb{E}_3^p(K) &\leqslant (1 - c^p\delta^2)\mathbb{E}_3^p(T^2), \end{aligned}$$

where c is a positive absolute constant. This estimate is asymptotically sharp as δ tends to zero.

3.3 Linear shadow systems and Steiner symmetrisation

For obtaining the stability versions of both the minimum and maximum inequalities, we shall use the following notion. Given a compact set Ξ in $\mathbb{R}^d$, a unit vector v, and for each $x \in \Xi$, a speed $\varphi(x) \in \mathbb{R}$, the corresponding *shadow system* is

$$\Xi_t = \{x + t\varphi(x)v : x \in \Xi\} \quad \text{for } t \in \mathbb{R}.$$

According to the classical work of H. Hadwiger [Had57], C.A. Rogers, G.C. Shephard [RS58] and Shephard [She64],

THEOREM 3.8 (Hadwiger,Rogers,Shephard). *For a shadow system Ξ_t, every quermass-integral of Ξ_t is a convex function of t.*

We note that for any $p \geqslant 1$, the convexity of the pth moment of the quermassintegrals follows as well.

In the last decades, shadow systems were successfully applied to various extremal problems about convex bodies (see e.g. S. Campi, P. Gronchi [CG06], and M. Meyer, Sh. Reisner [MR06]). For our purposes, we need a restricted class of shadow movements, introduced in [CCG99] by S. Campi, A. Colesanti, and P. Gronchi. We say that K_t, $t \in [a, b]$, is a *linear shadow system of convex bodies*, if we start with a convex body K, the speed $\varphi(x)$ is constant along any chord of K parallel to v, and

$$K_t = \{x + t\varphi(x)v : \, x \in K\} \quad \text{for } t \in [a, b]$$

is convex for every $t \in [a, b]$. In this case, $\varphi(x)$ is continuous on K, and it depends only on the projection $\pi_v x$ of x to $v^{\perp}$. Moreover, the volume of K_t is constant, and the transformation $x \mapsto x + t\varphi(x)v$ from K to K_t is measure preserving.

For any linear shadow system K_t, there also exists a linear shadow system $\widetilde{K_t}$, $t \in [a, b]$, such that

$$\gamma(\widetilde{K_t}) = o \text{ for } t \in [a, b], \text{ and each } \widetilde{K_t} \text{ is a translate of } K_t. \tag{3.14}$$

To see this, note that

$$\gamma(K_t) = \gamma(K) + t \cdot v \cdot V(K)^{-1} \int_K \varphi(z)\, dz. \tag{3.15}$$

Therefore, $\widetilde{K_t} = K_t - \gamma(K_t)$ can be achieved by using the speed

$$\tilde{\varphi}(x) = \varphi(x + \gamma(K)) - V(K)^{-1} \int_K \varphi(z)\, dz \quad \text{for } x \in \widetilde{K}.$$

The main reason for restricting shadow movements is the following result of [CCG99] (where linear shadow systems were called RS-movements).

THEOREM 3.9 (Campi, Colesanti, Gronchi). *If K_t, $t \in [a,b]$, is a linear shadow system, then $\mathbb{E}_n^p(K_t)$, $\mathbb{E}_o^p(K_t)$ and $\mathbb{E}_*^p(K_t)$ are convex functions of t. If either of these convex functions is linear, then any two elements of the system are affine images of each other, and actually linear images in the case of $\mathbb{E}_o^p(K_t)$.*

We note that although Theorem 3.9 was proved only for $\mathbb{E}_n^p(K_t)$ in [CCG99], the method works for the other functionals as well (see also Lemma 1 for a direct approach). Indeed, for handling $\mathbb{E}_n^p(K_t)$, the authors consider for each n-tuple $\Xi = \{x_1, \ldots, x_n\} \subset K$ the associated shadow system

$$\Xi_t = [x_1 + t\varphi(x_1)v, \ldots, x_n + t\varphi(x_n)v].$$

Since $V_d(\Xi_t)$ is a convex function of t by Theorem 3.8, we conclude Theorem 3.9 by

$$\begin{aligned}\mathbb{E}_n^p(K_t) &= V(K)^{-n-p} \times \\ &\int_K \cdots \int_K V([x_1 + t\varphi(x_1)v, \ldots, x_n + t\varphi(x_n)v])^p \, dx_1 \ldots dx_n.\end{aligned}$$

In order to obtain the convexity of $\mathbb{E}_o^p(K_t)$, to each d-tuple $\{x_1, \ldots, x_d\} \subset K \backslash o$ one assigns the $d+1$-tuple $\Xi = \{o, x_1, \ldots, x_d\}$, and defines the speed of o to be zero. The convexity $\mathbb{E}_*^p(K_t)$ follows from (3.14).

Finally, we have to deal with the extremal situations only. The argument is based on ideas in [CCG99]. Let us indicate it in the case when $\mathbb{E}_o^p(K_t)$ is a linear function of t, which also settles the case when $\mathbb{E}_*^p(K_t)$ is a linear function of t. If for some $s, t \in [a,b]$, $s < t$, K_t and K_s are not images of each other by any linear transformation, then there exist $\tau + \mu, \tau - \mu \in [s,t]$, $\mu > 0$, and d-tuple $\{x_1, \ldots, x_d\} \subset K$ with the property that $\{x_1 + \tau\varphi(x_1)v, \ldots, x_d + \tau\varphi(x_d)v\}$ is linearly dependent, and $\{x_1 + (\tau+\mu)\varphi(x_1)v, \ldots, x_d + (\tau+\mu)\varphi(x_d)v\}$ is linearly independent. It follows for $\Xi = \{x_1, \ldots, x_d, o\}$ that $V_d(\Xi_\tau) < \frac{1}{2}(V_d(\Xi_{\tau-\mu}) + V_d(\Xi_{\tau+\mu}))$, which in turn yields $\mathbb{E}_o^p(K_\tau) < \frac{1}{2}(\mathbb{E}_o^p(K_{\tau-\mu}) + \mathbb{E}_o^p(K_{\tau+\mu}))$ by Theorem 3.8 and the continuity of φ.

When dealing with linear shadow systems, the following simple observation is very useful. If $p > 0$, $\sigma_0, \ldots, \sigma_d$ are parallel segments, and Φ is an affine transformation that acts by translation along any line parallel to the σ_i's, then

$$\int_{\sigma_1} \cdots \int_{\sigma_d} V([o, z_1, \ldots, z_d])^p \, dz_1 \ldots dz_d$$
$$= \int_{\Phi\sigma_1} \cdots \int_{\Phi\sigma_d} V([\Phi o, \Phi z_1, \ldots, \Phi z_d])^p \, dz_1 \ldots dz_d, \tag{3.16}$$

and

$$\int_{\sigma_0} \cdots \int_{\sigma_d} V([z_0, \ldots, z_d])^p \, dz_0 \ldots dz_d$$
$$= \int_{\Phi\sigma_0} \cdots \int_{\Phi\sigma_d} V([\Phi z_0, \ldots, \Phi z_d])^p \, dz_0 \ldots dz_d. \tag{3.17}$$

All the known proofs of Theorem 3.2 use the fact that the moments to be estimated are monotone decreasing with respect to Steiner symmetrisation. This is a consequence Theorem 3.9, due to the following connection between Steiner symmetrals and shadow systems. Let K be a convex body, and H a hyperplane. Consider the unique linear shadow system K_t, $t \in [-1, 1]$, such that $K_1 = K$, and K_{-1} is the reflected image of K through H. Then K_0 is the Steiner symmetral K_H of K with respect to H. Now, Theorem 3.2 follows by using the well-known fact that $V(K)^{\frac{1}{d}} B^d$ can be obtained as a limit of a sequence of Steiner symmetrals starting from K.

The behaviour of $\mathbb{E}^p_{d+1}(K)$, $\mathbb{E}^p_o(K)$ and $\mathbb{E}^p_*(K)$ under Steiner symmetrisation can be computed easily using basic properties of determinants. Refining the proof, we will be able to deduce the stability estimates. It goes as follows. Assume that we take the Steiner symmetral of K with respect to H. Let $x_0, \ldots, x_d$ be an arbitrary set of points of H, and consider the integral over those simplices whose vertices project to the points (x_i) in H. By (3.16) and (3.17), we may assume that the midpoints of the chords of K through $x_0, \ldots, x_{d-1}$ are located in H. Then the Steiner symmetrisation moves only $\sigma(x_d)$, and the situation is easily handled.

For Lemmas 3.10 and 3.11, let $x_0, \ldots, x_d$ be contained in a hyperplane H in $\mathbb{R}^d$ in a way such that no d of them are contained in any $(d-2)$-plane, and let v be a unit vector not parallel to H. In addition, let $\delta > 0$, $\alpha_0 \geqslant 0$, and $\alpha_i > 0$ for $i = 1, \ldots, d$. For Lemma 3.10, to save space, we also use the (slightly obscure) convention that $\int_{J_0} dt_0 = 1$ for $J_0 = \{x_0\}$.

Lemma 3.10. *Let $p \geqslant 1$, let $0 \leqslant \beta_i < \alpha_i$ for $i = 1, \dots, d$, and let $\beta_0 = \alpha_0$, if $\alpha_0 = 0$, and $0 \leqslant \beta_0 < \alpha_0$ if $\alpha_0 > 0$. For $J_i = [-\alpha_i, -\beta_i] \cup [\beta_i, \alpha_i]$, $0 = 1, \dots, d$, we have*

$$\varphi(s) = \int_{J_d+s} \int_{J_{d-1}} \dots \int_{J_0} V([x_0 + t_0 v, \dots, x_d + t_d v])^p \, dt_0 \dots dt_d$$

is convex, and $\varphi(s) \geqslant \varphi(0)$.

Proof. For any fixed $t_i \in J_i$, $i = 0, \dots, d$, the function

$$V([x_0 + t_0 v, \dots, x_{d-1} + t_{d-1} v, x_d + (t_d + s)v])^p$$

of s is convex because it is the pth power of the absolute value of a linear function. Therefore $\varphi(s)$ is convex as well. Since $\varphi(s)$ is even, we have $\varphi(s) \geqslant \varphi(0)$. $\square$

Naturally, Lemma 3.10 with $\beta_i = 0$, $i = 0, \dots, d$, directly yields Theorem 3.9 for $\mathbb{E}^p_{d+1}(K)$, $\mathbb{E}^p_o(K)$ and $\mathbb{E}^p_*(K)$. Now we provide a stability version under a technical (but necessary) side condition.

Lemma 3.11. *Let $p \geqslant 1$ and $\delta \in (0, \alpha_d/2)$, and assume that if $|t_i| \leqslant \alpha_i$ for every $i = 0, \dots, d-1$, then*

$$\operatorname{aff}\{x_0 + t_0 v, \dots, x_{d-1} + t_{d-1} v\} \cap [x_d - (\alpha_d - \delta)v, x_d + (\alpha_d - \delta)v] \neq \emptyset. \tag{3.18}$$

Then the following inequalities hold.

(i) In the case $\alpha_0 = 0$:

$$\begin{aligned} &\int_{-\alpha_d+\delta}^{\alpha_d+\delta} \int_{-\alpha_{d-1}}^{\alpha_{d-1}} \dots \int_{-\alpha_1}^{\alpha_1} V([x_0, x_1 + t_1 v, \dots, x_d + t_d v])^p \, dt_1 \dots dt_d \\ &\qquad - \int_{-\alpha_d}^{\alpha_d} \int_{-\alpha_{d-1}}^{\alpha_{d-1}} \dots \int_{-\alpha_1}^{\alpha_1} V([x_0, x_1 + t_1 v, \dots, x_d + t_d v])^p \, dt_1 \dots dt_d \\ &\geqslant \delta^2 \, \frac{p 2^{d-p-1}}{d^p} \, \alpha_1 \dots \alpha_{d-1} \alpha_d^{p-1} \, V_{d-1}(\pi_v[x_0, \dots, x_{d-1}])^p. \end{aligned}$$

(ii) If $\alpha_0 > 0$, then

$$\begin{aligned} &\int_{\delta-\alpha_d}^{\delta+\alpha_d} \int_{-\alpha_{d-1}}^{\alpha_{d-1}} \dots \int_{-\alpha_0}^{\alpha_0} V([x_0 + t_0 v, \dots, x_d + t_d v]) \, dt_0 \dots dt_d \\ &\qquad - \int_{-\alpha_d}^{\alpha_d} \int_{-\alpha_{d-1}}^{\alpha_{d-1}} \dots \int_{-\alpha_0}^{\alpha_0} V([x_0 + t_0 v, \dots, x_d + t_d v])^p \, dt_0 \dots dt_d \\ &\geqslant \delta^2 \, \frac{p 2^{d-p}}{d^p} \, \alpha_0 \dots \alpha_{d-1} \alpha_d^{p-1} \, V_{d-1}(\pi_v[x_0, \dots, x_{d-1}])^p. \end{aligned}$$

Proof. We prove only (ii); obtaining (i) by the same method is straightforward. Due to condition (3.18) and symmetry, and by using the notation

$$\begin{aligned}\omega(t_0,t_1,\ldots,t_d) =& V([x_0+t_0v,\ldots,x_{d-1}+t_{d-1}v,x_d+t_dv])^p\\ &+V([x_0-t_0v,\ldots,x_{d-1}-t_{d-1}v,x_d+t_dv])^p,\end{aligned}$$

the following holds:

$$\begin{aligned}&2\int_{-\alpha_d+\delta}^{\alpha_d+\delta}\int_{-\alpha_{d-1}}^{\alpha_{d-1}}\cdots\int_{-\alpha_0}^{\alpha_0}V([x_0+t_0v,\ldots,x_d+t_dv])^p\,dt_0\ldots dt_d\\ &\quad-2\int_{-\alpha_d}^{\alpha_d}\int_{-\alpha_{d-1}}^{\alpha_{d-1}}\cdots\int_{-\alpha_0}^{\alpha_0}V([x_0+t_0v,\ldots,x_d+t_dv])^p\,dt_0\ldots dt_d\\ &=\int_{\alpha_d-\delta}^{\alpha_d}\cdots\int_{-\alpha_0}^{\alpha_0}\omega(t_0,\ldots,t_{d-1},t_d+\delta)-\omega(t_0,\ldots,t_{d-1},t_d)\,dt_0\ldots dt_d.\end{aligned}$$

For fixed $t_i\in[-\alpha_i,\alpha_i]$, $i=0,\ldots,d-1$ and $t_d\in[\alpha_d-\delta,\alpha_d]$, let $s\in[-\alpha_d+\delta,\alpha_d-\delta]$ satisfy that $x_0+t_0v,\ldots,x_{d-1}+t_{d-1}v$ and x_d+sv are contained in a hyperplane. It follows that

$$\begin{aligned}\omega(t_0,\ldots,t_{d-1},t_d+\delta)-\omega(t_0,\ldots,t_{d-1},t_d)=\\ \frac{V_{d-1}(\pi_v[x_0,\ldots,x_{d-1}])^p}{d^p}\times\\ [(t_d+\delta+s)^p+(t_d+\delta-s)^p-(t_d+s)^p-(t_d-s)^p].\end{aligned}$$

We claim that

$$(t_d+\delta+s)^p+(t_d+\delta-s)^p-(t_d+s)^p-(t_d-s)^p\geqslant p\delta\alpha_d^{p-1}/2^{p-1}. \tag{3.19}$$

We may assume that $s\geqslant 0$, and hence $s\in[0,t_d]$. Let $\psi(s)$ be the left hand side of (3.19) as a function of s, then

$$\psi'(s)=p(t_d+\delta+s)^{p-1}-p(t_d+\delta-s)^{p-1}-[p(t_d+s)^{p-1}-p(t_d-s)^{p-1}].$$

Since $p\tau^{p-1}$ is convex, if $p\geqslant 2$, and concave, if $1\leqslant p<2$ for $\tau>0$, we deduce that ψ' is non-negative, hence ψ is increasing, if $p\geqslant 2$, and ψ' is non-positive, hence ψ is decreasing, if $1\leqslant p<2$. In particular, we may assume $s=0$, if $p\geqslant 2$, and $s=t_d$, if $1\leqslant p<2$ in (3.19). Therefore the estimates $t_d\geqslant\alpha_d/2$ and $(\tau+\delta)^p-\tau^p>p\delta\tau^{p-1}$ for $\tau=t_d$ or $\tau=2t_d$ yield (3.19). In turn we conclude Lemma 3.11. □

3.4 Stability of the minimum inequalities

We are going to use Vinogradov's $\gg$ notation in the following sense: $f \gg g$ or $g \ll f$ for non-negative functions f and g iff there exists a constant $c > 0$ depending only on d, for which $f \geqslant cg$ holds. In addition, we write $h = O(f)$ if $|h| \ll f$.

We will say that a convex body $K \subset \mathbb{R}^d$ is in *John position*, if its unique inscribed ellipsoid of maximal volume is B^d. We are going to use the following simple consequence of Fritz John's ellipsoid theorem (see [Joh48] and [Bal97]).

Proposition 3.12. *Assume that the o-symmetric convex body $K \subset \mathbb{R}^d$ is in John position. Then for any point $p \in S^{d-1}$, there is a contact point q between K and B^d, for which $\langle p, q\rangle \geqslant 1/\sqrt{d}$.*

The statement is equivalent to the well-known fact that any point in K has norm at most $\sqrt{d}$.

We will use the following notations. Let K be a convex body in R^d. Let H be a hyperplane of R^d with normal v. Let ℓ be the line of direction v, and for any $x \in H$, denote by $\sigma(x)$ the secant $K \cap (x + \ell)$, and by $M(x)$ the midpoint of $\sigma(x)$. Moreover, let $m(x)$ be the signed distance of x and $M(x)$, that is, $m(x) = \langle M(x) - x, v\rangle$.

Now, for Theorem 3.6. First, we deal with the case when K is o-symmetric and its Banach-Mazur distance from B^d is sufficiently small. This is the core of the proof.

Lemma 3.13. *For any $d \geqslant 2$, there exists $\varepsilon_0, \hat{\gamma} > 0$, such that if $K \subset \mathbb{R}^d$ is an o-symmetric convex body in John position, and the maximal norm of the points of K is $1 + \varepsilon$ with $\varepsilon \leqslant \varepsilon_0$, then for any $p \geqslant 1$,*

$$\mathbb{E}_o^p(K) - \mathbb{E}_o^p(B^d) \geqslant \hat{\gamma}^p \varepsilon^{(d+3)/2}, \text{ and}$$
$$\mathbb{E}_{d+1}^p(K) - \mathbb{E}_{d+1}^p(B^d) \geqslant \hat{\gamma}^p \varepsilon^{(d+3)/2}.$$

Proof. Let r be a point of K of maximal norm. By Proposition 3.12, there is a contact point $q \in \partial K \cap S^{d-1}$ with $\langle -r, q\rangle \geqslant \|r\|/\sqrt{d}$. Let ℓ be the line passing through r, q with direction vector $v = (r - q)/\|r - q\|$, let $H = v^\perp$, and choose a coordinate system such that the dth coordinate axis is parallel to ℓ. Taking $x_d = \pi_v r = \pi_v q$, a simple calculation shows that

$$\|x_d\| < \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{d}}\,. \tag{3.20}$$

For any $x \in H \cap B^d$, let $\sigma(x) = K \cap (x+\ell)$ with midpoint $M(x)$, and define $m(x) = \langle (M(x)-x), v\rangle$. Since $B^d \subset K \subset (1+\varepsilon)B^d$, if $\|x\| \leqslant 0.9$, then $m(x)$ can be estimated as

$$|\,m(x)| \leqslant \frac{\sqrt{(1+\varepsilon)^2 - \|x\|^2} - \sqrt{1-\|x\|^2}}{2} = \frac{\varepsilon(1+O(\varepsilon))}{2\sqrt{1-\|x\|^2}}. \tag{3.21}$$

Note that for $x = x_d$, equality holds in (3.21).

The estimating function is illustrated on Figure 3.1.

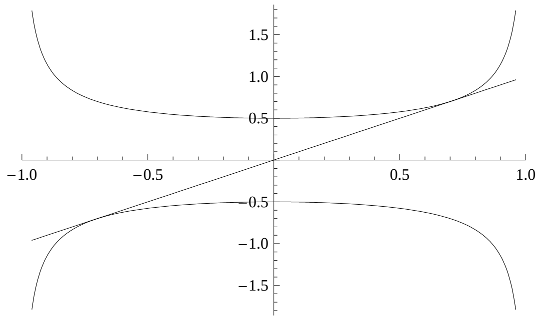

Figure 3.1: Estimating the deviation

The tangent from o to the graph of $f(z) = 1/\sqrt{1-z^2}$ has its contact point at $z = 1/\sqrt{2}$. Due to the convexity of $f(z)$, estimates (3.20) and (3.21) imply that if we choose the points $x_1, \ldots, x_{d-1}$ of norm about $1/\sqrt{2}$ with $x_d \in [o, x_1, \ldots, x_{d-1}]$, then $M(x_d)$ is separated from $[o, M(x_1), \ldots, M(x_d)]$ by $c\varepsilon$, where c is a constant depending on d only. This then yields a positive error $\mathbb{E}_o^p(K)$ in comparison with $\mathbb{E}_o^p(B^d)$. This idea is transformed to a quantitative proof as follows.

First, we estimate the decay of $m(x)$ around x_d. By convexity, $[B^d, r\,] \subset K$. Let $\hat{r} = S^{d-1} \cap [o\,, r\,]$, and $\tilde{r} = S^{d-1} \cap ([q\,r\,] \setminus q)$. Estimate (3.20) yields that $\|\hat{r} - \tilde{r}\| \leqslant \varepsilon$. For $s \in S^{d-1}$, denote by $T(s)$ be the tangent hyperplane to S^{d-1} at s. It is easily obtained that the intersection $[B^d, r\,] \cap T(\hat{r})$ is a $(d-1)$-dimensional ball of radius $\sqrt{\varepsilon/(2+\varepsilon)}$, and thus, $A = [B^d, r\,] \cap T(\tilde{r})$ contains a ball of radius $\sqrt{\varepsilon/2.5}$ centred at $\tilde{r}$. Then, again by (3.20), $\pi_v(A)$ contains a ball D of radius $\sqrt{\varepsilon}/4$ centred at x_d. Since $T(q)$ is a tangent hyperplane of K, $m(x)$ can be estimated over D linearly:

$$m\left(x_d + t\frac{\sqrt{\varepsilon}}{4}u\right) \geqslant (1-t)m(x_d),\ \forall\, u \in S^{d-1}, \forall\, t \in [0,1]. \tag{3.22}$$

Next, we are going to estimate $\mathbb{E}_o^p(K) - \mathbb{E}_o^p(B^d)$. Let $x_0 = o$, and choose $x_1, \ldots, x_{d-1}$ as follows. Take $\tilde{y} = x_d/\|x_d\|$. If $d = 2$, then let $x_1 = \tilde{y}/\sqrt{2}$. If $d \geqslant 3$, then take $y = (1/\sqrt{2} - 1/(100d))\tilde{w}$, and let $x_1, \ldots, x_{d-1}$ be of norm $1/\sqrt{2} - 1/(500d)$, the vertices of a regular $(d-2)$-simplex in $(y+y^\perp)\cap H$ with centroid y. Note that the distance between any two of these is $> 1/100\sqrt{d}$. Let $\varrho = 1/(1000d)$ and define $X_i = x_i + \varrho B^{d-1} \subset H$ for every $i = 1, \ldots, d-1$. Then $V_{d-1}(X_i) \gg 1$.

Note that by (3.20), there exists a neighbourhood U of x_d of radius $\gg \sqrt{\varepsilon}$ in H such that for any $x_i' \in X_i$, $i = 1, \ldots, d-1$, we have $U \subset [o, x_1', \ldots, x_{d-1}']$. For such a collection of (x_i'), and for any $x_d' \in U$, define

$$D((x_i')) = D(x_1', \ldots, x_{d-1}', x_d') = [o, M(x_1'), \ldots, M(x_{d-1}')] \cap \sigma(x_d'),$$

and let $d((x_i')) = \langle D((x_i')), v\rangle$. Note that for any $x_i' \in X_i$, $1 \leqslant i \leqslant d-1$,

$$\frac{1}{\sqrt{2}} - \frac{3}{500d} \leqslant \|x_i'\| \leqslant \frac{1}{\sqrt{2}} - \frac{1}{500d}.$$

Thus, (3.21) yields that there exists a neighbourhood $V \subset U$ of x_d in H, still of area $\gg \varepsilon^{(d-1)/2}$, such that for any $x_i' \in X_i$, $i = 1, \ldots, d-1$ and any $x_d' \in V$, for sufficiently small ε we have

$$d(x_1', \ldots, x_d') \leqslant \frac{\|x_d\|\varepsilon}{1 - 1/(100d)}. \tag{3.23}$$

Since

$$\frac{\varepsilon}{2\sqrt{1 - \|x_d\|^2}} - \frac{\|x_d\|\varepsilon}{1 - 1/(100d)}$$

as a function of $\|x_d\|$ is decreasing for $\|x_d\| < 1/\sqrt{2}$, estimates (3.20), (3.21) and (3.23) yield that for $x_i' \in X_i$ and $x_d' \in V$,

$$\begin{aligned} m(x_d) - d(x_1', \ldots, x_d') &\geqslant \varepsilon \left(\frac{1}{\sqrt{2} + 1/(2\sqrt{d})} - \frac{1/\sqrt{2} - 1/(4\sqrt{d})}{1 - 1/(100d)} \right) \\ &\geqslant \frac{\varepsilon}{20d}\ . \end{aligned}$$

Let $R = \sqrt{2\varepsilon}/(100d)$, and take $X_d = V \cap (x_d + RB^{d-1}) \subset H$. Then $V_{d-1}(X_d) \gg \varepsilon^{(d-1)/2}$. Moreover, since $m(x_d) < \varepsilon/\sqrt{2}$, the above estimate and (3.22) yield that for $x_i' \in X_i$, $i = 1, \ldots, d$,

$$m(x_d') - d((x_1', \ldots, x_d')) \geqslant \frac{\varepsilon}{100d}. \tag{3.24}$$

Let now K' be the Steiner symmetral of K with respect to H. By Theorem 3.9, it is sufficient to prove that $\mathbb{E}_o^p(K) - \mathbb{E}_o^p(K') \geqslant \hat{\gamma}^p \varepsilon^{(d+3)/2}$. We calculate the average volume of random simplices by integrating along the d-tuples of chords of K parallel to v. For $x \in H$, let $\sigma_K(x) = \sigma(x) = K \cap (x + \ell)$, and $\sigma_{K'}(x) = K' \cap (x + \ell)$. For

$x'_1, \dots, x'_d \in H \cap K$, define

$$\omega(x'_1, \dots, x'_d) = \int_{\sigma_K(x'_1)} \cdots \int_{\sigma_K(x'_d)} V[o, y_1, \dots, y_d]\, dy_d \dots dy_1 \\ - \int_{\sigma_{K'}(x'_1)} \cdots \int_{\sigma_{K'}(x'_d)} V[o, y_1, \dots, y_d]\, dy_d \dots dy_1$$

Lemma 3.10 yields that for any $(x'_i)_1^d \subset H \cap K$, we have $\omega(x'_1, \dots, x'_d) \geqslant 0$. Moreover, by the construction of $(X_i)_1^{d-1}$, for any $x'_i \in X_i$, we have $V_{d-1}([o, x'_1, \dots, x'_{d-1}]) \gg 1$. Thus, by (3.24), Lemma 3.10, and part (i) of Lemma 3.11,

$$\mathbb{E}_o^p(K) - \mathbb{E}_o^p(K') \geqslant \int_{X_1} \cdots \int_{X_d} \omega(x'_1, \dots, x'_d)\, dx'_d \dots dx'_1 \geqslant \gamma_1^p \varepsilon^{(d+3)/2}$$

for some $\gamma_1 > 0$ depending only on d.

Next, we estimate $\mathbb{E}_{d+1}^p(K) - \mathbb{E}_{d+1}^p(K')$. We start as before. There are two cases to be considered depending on $\|x_d\|$. First, assume that $\|x_d\| \geqslant 1/100$ (we need only $\|x_d\| \gg 1$). Then construct $(X_i)_1^d$ as before. Choose $R > 0$ small enough such that the following hold:

i) For any x'_0 with $\|x'_0\| \leqslant R$, and any $x'_i \in X_i$, $i = 1, \dots, d$, we have $x'_d \in [x'_0, \dots, x'_{d-1}]$

ii) For any x'_0 with $\|x'_0\| \leqslant R$ and $m(x'_0) \leqslant 0$, and any $x'_i \in X_i$, $i = 1, \dots, d$,

$$m(x'_d) - \langle [M(x'_0), \dots, M(x'_{d-1})] \cap \sigma(x'_d), v \rangle \gg \varepsilon. \qquad (3.25)$$

Let $X_0 = \{x \in H : |x| < R,\ m(x) \leqslant 0\}$. By the symmetry of K, the measure of X_0 is at least half as large as that of RB^{d-1}, thus, $V_{d-1}(X_0) \gg 1$. Then, part (ii) of Lemma 3.11 applies as before, yielding

$$\mathbb{E}_{d+1}^p(K) - \mathbb{E}_{d+1}^p(K') \geqslant \gamma_2^p \varepsilon^{(d+3)/2}$$

for some $\gamma_2 > 0$ depending only on d.

In the second case, x_d is close to the origin: $\|x_d\| < 1/100$. Let A be the annulus $\{x \in H : 1/2 < \|x\| < 3/4\}$. For this instance, define the function d' on A^d by

$$d'(x'_0, \dots, x'_{d-1}) = \langle ([M(x'_0), \dots, M(x'_{d-1})] \cap \sigma(o)), v \rangle.$$

Note that by symmetry, $d'(-x'_0, \ldots, -x'_{d-1}) = -d'(x'_0, \ldots, x'_{d-1})$. Let $C = 1/100$, and consider only those $(x'_i)_0^{d-1} \subset A$, for which $|\langle u, v\rangle| \geqslant C$, where u is the normal vector of $[M(x'_0), \ldots, M(x'_{d-1}))]$. Then the (product) measure of these point sets is $\gg 1$; moreover, at least half of them satisfies $d'(x'_0, \ldots, x'_{d-1}) \leqslant 0$. These also satisfy (3.25). Thus, integrating over these sets, the argument works as before. □

Remark. In the planar case, one can obtain the following quantitative result: If K satisfies the conditions of Lemma 3.11, then for small $\varepsilon > 0$,

$$\mathbb{E}_o^1(K) - \mathbb{E}_o^1(B^2) > \frac{\varepsilon^{5/2}}{400}\,.$$

From this, it also follows that if K is a centrally symmetric convex disc, and $\mathbb{E}_o^1(K) \leqslant (1+\delta)\mathbb{E}_o^1(B^2)$, then there exists an ellipse E, for which $E \subset K \subset (1 + 20\delta^{2/5})E$.

To obtain the estimate for not necessarily symmetric bodies, we cite the following result of K. J. Böröczky, see Theorem 1.4 of [Bör].

Lemma 3.14. *For any convex body $K \subset \mathbb{R}^d$ with $\delta_{\mathrm{BM}}(K, B^d) > 1 + \varepsilon$ for some $\varepsilon > 0$, there exists an o-symmetric convex body C with axial rotational symmetry and a constant $\gamma > 0$ depending only on d, such that $\delta_{\mathrm{BM}}(C, B^d) > \gamma\varepsilon^2$, and C results from K as a limit of subsequent Steiner symmetrisations and affine transformations.*

Now, we are ready to prove the general result.

Proof of Theorem 3.6. Let $\delta_{\mathrm{BM}}(K, B^d) = 1 + \delta$. By Lemma 3.14, we may assume that K is an o-symmetric convex body in John position, provided we prove

$$\mathbb{E}_o^p(K) \geqslant (1 + \gamma^p \delta^{\frac{d+3}{2}})\mathbb{E}_o^p(B^d), \text{ and} \tag{3.26}$$

$$\mathbb{E}_{d+1}^p(K) \geqslant (1 + \gamma^p \delta^{\frac{d+3}{2}})\mathbb{E}_{d+1}^p(B^d) \tag{3.27}$$

for $\gamma > 0$ depending only on d. Let the maximal norm of points of K be $1 + \varepsilon$. Since the volume of $K \setminus B^d$ is $\gg \varepsilon^{(d+1)/2}$, it follows that

$$\gamma_0 \varepsilon^{\frac{d+1}{2}} \leqslant \delta \leqslant \varepsilon \tag{3.28}$$

for $\gamma_0 > 0$ depending only on d.

Let ε_0 and $\hat\gamma$ come from Lemma 3.13. If $\delta \leqslant \delta_0 = \gamma_0 \varepsilon_0^{\frac{d+1}{2}}$ then $\varepsilon \leqslant \varepsilon_0$ by (3.28), and hence we have (3.26) and (3.27) with $\gamma = \hat\gamma$ by Lemma 3.13 and (3.28).

Therefore we may assume that $\delta > \delta_0$. Choose a sequence of Steiner symmetrals $K_0, K_1, K_2, \dots$ starting with $K = K_0$ that converge to B^d, and hence there exists K_n such that $\delta_{BM}(K_{n+1}) \leqslant \delta_0 < \delta_{BM}(K_n)$. Let $L_t : t \in [-1, 1]$ be the linear shadow system with $L_1 = K_n$ and $L_0 = K_{n+1}$ corresponding to the Steiner symmetrisation of K_n (see Section 3.3), thus there exists $t \in [0, 1)$ such that $\delta_{BM}(L_t) = \delta_0$. It follows that $\mathbb{E}^p_*(L_t) \leqslant \mathbb{E}^p_*(K_n) \leqslant \mathbb{E}^p_*(K)$ and $\mathbb{E}^p_{d+1}(L_t) \leqslant \mathbb{E}^p_{d+1}(K_n) \leqslant \mathbb{E}^p_{d+1}(K)$, thus we conclude (3.26) and (3.27) by the previous case and $\delta < \sqrt{d}$. □

We made no attempt to find the best possible constants. However, the estimate $\varepsilon^{(d+3)/2}$ for centrally symmetric K is close to the truth: if $K = [r, -r, B^d]$, where r is of norm $1 + \varepsilon$, then

$$\frac{\mathbb{E}^p_o(K)}{\mathbb{E}^p_o(B^d)} - 1 \ll \varepsilon^{(d+1)/2}.$$

3.5 Stability of the maximum inequalities in the plane

Since here we work only on the plane, a *convex disc* means a planar convex body, and $A(K) = V_2(K)$ is the area of K. For a polygon Π with at least four vertices $q_1, \dots, q_k$ in this order, a *basic linear shadow system* at q_1, basic system for short, is defined as follows. Let q_1' and q_1'' be points different from q_1 such that $q_1 \in [q_1', q_1'']$, $q_1' - q_1''$ is parallel to $q_2 - q_k$, and $q_2, \dots, q_k$ lie on the boundary of $\Pi' = [q_1', q_2, \dots, q_k]$ and $\Pi'' = [q_1'', q_2, \dots, q_k]$. The corresponding basic system is the unique linear shadow system Π_t, $t \in [-\beta, \alpha]$, such that $\alpha, \beta > 0$, $\alpha + \beta = 1$, $\Pi_{-\beta} = \Pi''$, $\Pi_0 = \Pi$, and $\Pi_\alpha = \Pi'$. In this case, the generating vector is parallel to $q_2 - q_k$, and the speed of any point in $[q_2, \dots, q_k]$ is zero. It follows from Theorem 3.9 that for any $n \geqslant 3$ and $p \geqslant 1$,

$$\mathbb{E}^p(\Pi) < \max\{\mathbb{E}^p(\Pi'), \mathbb{E}^p(\Pi'')\}, \tag{3.29}$$

where $\mathbb{E}^p(\Pi)$ stands either for $\mathbb{E}^p_o(\Pi)$ or $\mathbb{E}^p_n(\Pi)$. More precisely, the following holds:

$$\mathbb{E}^p(\Pi_{-t}) \text{ on } [0, \beta], \text{ or } \mathbb{E}^p(\Pi_t) \text{ on } [0, \alpha], \text{ is strictly increasing.} \tag{3.30}$$

For a convex disc K, let T_K be a triangle of maximal area contained in K. It follows that the triangle, the midpoints of whose sides are the vertices of T_K, contains K. In particular, $A(K) < 4A(T_K)$.

First, we reduce the case to polygons with at most 6 vertices.

Proposition 3.15. *For a convex disc K, let $\widetilde{T}$ be the triangle, the midpoints of whose sides are the vertices of T_K. For $n \geqslant 3$ and $p \geqslant 1$, there exist polygons Π_1 and Π_2 with $A(\Pi_1) = A(\Pi_2) = A(K)$ such that $T_K \subset \Pi_1, \Pi_2$, all vertices of Π_1, Π_2 are on $\partial\widetilde{T}$, and*

$$\mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(\Pi_1) \quad \textit{and} \quad \mathbb{E}_*^p(K) \leqslant \mathbb{E}_*^p(\Pi_2).$$

Proof. We may assume that K is not a triangle, and by continuity, that K is a polygon. However, for $k \geqslant 4$, suitable basic systems and (3.29) yield that among polygons P of at most k vertices with fixed area such that $T_K \subset P \subset \widetilde{T}$, any polygon maximising either $\mathbb{E}_n^p(P)$ or $\mathbb{E}_*^p(P)$ has all of its vertices in $\partial\widetilde{T}$. □

The core lemma comes.

Lemma 3.16. *There exist positive absolute constants $\varepsilon_0, \hat{c}$ such that if $p \geqslant 1$, and $A(K) = (1+\varepsilon)A(T_K)$ for a convex disc K and $\varepsilon \in (0, \varepsilon_0]$, then*

$$\mathbb{E}_3^p(K) \leqslant (1 - \hat{c}^p\varepsilon^2)\mathbb{E}_3^p(T^2) \quad \textit{and} \quad \mathbb{E}_*^p(K) \leqslant (1 - \hat{c}^p\varepsilon^2)\mathbb{E}_*^p(T^2).$$

Proof. We first consider $\mathbb{E}_*^p(K)$. Let $T_K = [p_1, p_2, p_3]$, and let q_1, q_2, q_3 be the such that p_i is the midpoint of $[q_j, q_k]$, $\{i, j, k\} = \{1, 2, 3\}$. We may assume that each side of T_K is of length one, and $\gamma(T_K) = o$.

Let Π be the polygon provided by Claim 3.15, and let x be the farthest vertex of Π from T_K. We may assume that $x \in [p_1, q_2]$. It follows that $A([x, p_1, p_3])$ is between $\varepsilon A(T_K)/6$ and $\varepsilon A(T_K)$, and hence

$$\varepsilon/6 \leqslant \|x - p_1\| \leqslant \varepsilon. \tag{3.31}$$

Let us number the vertices of Π in such a way that $x = x_3$, its neighbouring vertices are $x_2 \in [p_1, q_3]$ and $x_4 \in [p_3, q_2]$, and the other neighbours of x_2 and x_4 are $x_1 \in [p_2, q_3]$ and x_5, respectively; see Figure 3.2. Here possibly $x_5 = x_1$, and either $x_5 \in [p_3, q_1]$, or $x_5 \in [p_2, q_1]$. The definition of $x = x_3$ yields that for any $i = 1, 2, 4, 5$ there exists $j \in \{1, 2, 3\}$ such that

$$\|x_i - p_j\| \leqslant \|x - p_1\|. \tag{3.32}$$

To deform Π, let l be the line parallel to $x_2 - x_4$ passing through x, and let x' and x'' be the intersections of l with $\mathrm{aff}\{x_2, x_1\}$ and $\mathrm{aff}\{x_4, x_5\}$, respectively. We consider the basic system Π_t, $t \in [-\beta, \alpha]$, $\alpha, \beta > 0$, $\alpha + \beta = 1$, where $\Pi_0 = \Pi$, x' is a vertex of Π_α, and x'' is a vertex of $\Pi_{-\beta}$. We write $\varphi(z)$ to denote the speed of a $z \in \Pi$, and observe that the generating vector is $v = \frac{x_2 - x_4}{\|x_2 - x_4\|}$.

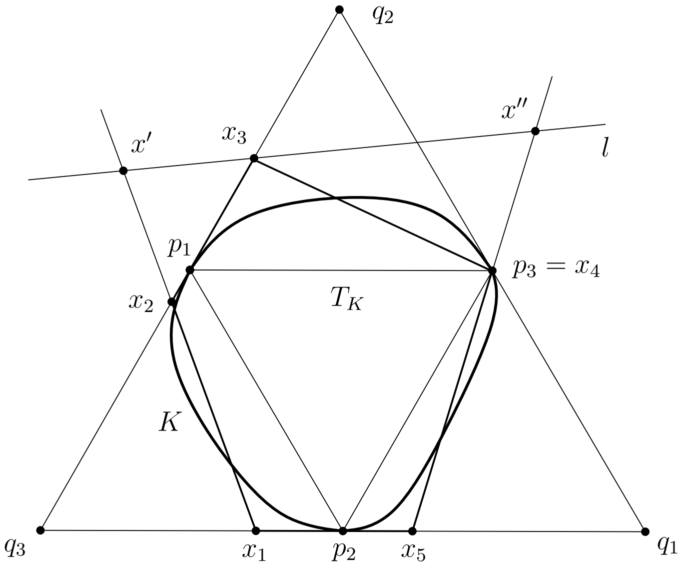

Figure 3.2: Modifying convex discs

It follows by (3.15) that $\gamma(\Pi) = \beta\,\gamma(\Pi_\alpha) + \alpha\,\gamma(\Pi_{-\beta})$. Thus for any $z_1, z_2 \in \Pi$, Theorem 3.8 yields

$$A([\gamma(\Pi), z_1, z_2])^p \leqslant \beta A([\gamma(\Pi_\alpha), z_1 + \alpha\varphi(z_1)v, z_2 + \alpha\varphi(z_2)v])^p$$
$$+ \alpha A([\gamma(\Pi_{-\beta}), z_1 - \beta\varphi(z_1)v, z_2 - \beta\varphi(z_2)v])^p. \quad (3.33)$$

In order to obtain a stability statement, we improve on (3.33). As a first step, we localise $\gamma(\Pi_t)$. The centroid $\gamma(\Pi)$ has the property that $-1/3\,(\Pi - \gamma(\Pi)) \subset \Pi - \gamma(\Pi)$. It follows by (3.31) and (3.32) that

$$\gamma(\Pi) \in 2\varepsilon T_K. \quad (3.34)$$

We note that by (3.32), $|\varphi(z)| \leqslant 1.1$ for $z \in \Pi$, and $\varphi(z) = 0$ if z is separated from $x = x_3$ by the diagonal $[x_2, x_4]$. Thus (3.15) yields

$$\gamma(\Pi_t) = \gamma(\Pi) + t\omega v, \quad \text{for } \omega \in (0, 2\varepsilon) \text{ independent of } t. \quad (3.35)$$

As $[p_1, p_3]$ is close to l (any $z \in [p_1, p_2]$ is of distance at most 3ε from l) and $[p_1, p_2]$ is close to $[x_2, x_1]$, we may choose ε_0 small enough to ensure $\varepsilon/12 \leqslant \|x - x'\| \leqslant 2\varepsilon$.

In addition, $[x_4, x_5]$ is either contained in $[q_2, q_1]$, or it is close to $[p_3, p_2]$, therefore $2/3 \leqslant \|x - x''\| \leqslant 3/2$. We deduce

$$\varepsilon/24 \leqslant \alpha \leqslant 4\varepsilon. \tag{3.36}$$

We may assume that $\mathbb{R} = v^{\perp}$, oriented in a way such that $\pi_v p_3 > 0$. We observe that

$$\frac{1}{2\sqrt{3}} - \varepsilon < \pi_v p_3 \leqslant \frac{1}{2\sqrt{3}} \leqslant \pi_v p_1 < \pi_v x_3 - \frac{\varepsilon}{12}.$$

For $y \in \pi_v \mathrm{int}\Pi$ and $t \in [-\beta, \alpha]$, we write $\sigma_t(y)$ to denote the chord of Π_t parallel to v and projecting into y, and $m_t(y)$ to denote the midpoint of $\sigma_t(y)$. In particular, $\sigma_t(y) = \sigma_0(y)$ if $y \leqslant \pi_v x_2$. If ε_0 is small enough then for any $s \in (0, \frac{1}{8})$,

$$|\langle v, m(\pi_v p_3 - s)\rangle| \leqslant 2s \text{ and } V_1(\sigma(y)) > \tfrac{\sqrt{3}}{2}.$$

We consider the intervals

$$I_1 = [\pi_v p_3 - \tfrac{1}{16}, \pi_v p_3 - \tfrac{1}{32}] \text{ and } I_2 = [\tfrac{1}{16}\pi_v p_1 + \tfrac{15}{16}\pi_v x_3, \tfrac{1}{32}\pi_v p_1 + \tfrac{31}{32}\pi_v x_3],$$

and hence (3.31) yields

$$V_1(I_1) = \tfrac{1}{32} \text{ and } V_1(I_2) \geqslant \tfrac{\varepsilon}{12\cdot 64}. \tag{3.37}$$

In addition, $\sigma_t(y) = \sigma_0(y)$ if $y \in I_1$ and $t \in [-\beta, \alpha]$. To ensure the condition (3.18) in Lemma 3.11, for $y \in I_1$, we restrict our attention to

$$\sigma_t^*(y) = \tfrac{1}{8}(\sigma_t(y) - m_t(y)) + m_t(y).$$

Our main claim is that there exists an absolute constant $c_1 > 0$, such that for any $y_1 \in I_1$ and $y_2 \in I_2$, the integral

$$f(t) = \int_{\sigma_t^*(y_1)} \int_{\sigma_t(y_2)} A([\gamma(\Pi_t), z_1, z_2])^p \, dz_1 dz_2$$

satisfies

$$\alpha f(-\beta) + \beta f(\alpha) \geqslant f(0) + c_1^p \varepsilon. \tag{3.38}$$

It follows by (3.34) and (3.35) that if ε_0 is small enough, then there exists a $\tau \in (\frac{1}{4}, \frac{3}{4})$, such that $\gamma(\Pi_{-\tau})$, $m_{-\tau}(y_1)$ and $m_{-\tau}(y_2)$ are collinear. Writing ω_t to denote the intersection point of $\mathrm{aff}\{\gamma(\Pi_t), m_t(y_1)\}$ and $\mathrm{aff}\sigma_t(y_2)$, the function $\langle v, \omega_t - m_t(y_2)\rangle$ of t is linear, zero at $-\tau$, and satisfies

$$\langle v, \omega_\alpha - m_\alpha(y_2)\rangle \geqslant \tfrac{1}{8} \text{ and } \langle v, \omega_{-\beta} - m_{-\beta}(y_2)\rangle \leqslant -\tfrac{1}{8}.$$

We deduce by Lemma 3.10 and (3.16) that $f(t)$ is convex, and has its minimum at $-\tau$. Thus Lemma 3.11 yields

$$f(\alpha), f(-\beta) \geqslant f(-\tau) + c_2^p \text{ for an absolute constant } c_2 > 0.$$

It follows by $\beta = 1 - \alpha$ and (3.36) that

$$\begin{aligned} \alpha f(-\beta) + \beta f(\alpha) - f(0) &\geqslant \alpha f(-\beta) + \beta f(\alpha) - \tfrac{\alpha}{\alpha+\tau} f(-\tau) - \tfrac{\tau}{\alpha+\tau} f(\alpha) \\ &= \alpha f(-\beta) + \alpha \cdot \tfrac{1-\alpha-\tau}{\alpha+\tau} f(\alpha) - \tfrac{\alpha}{\alpha+\tau} f(-\tau) \\ &\geqslant \tfrac{\alpha}{\alpha+\tau} \cdot c_2^p \geqslant \tfrac{c_2^p}{24} \cdot \varepsilon. \end{aligned}$$

Therefore we have verified (3.38). In turn combining this with (3.33) and (3.37) proves for a suitable absolute constant $c_3 > 0$, that

$$\mathbb{E}_*^p(\Pi) + c_3^p \varepsilon^2 \leqslant \beta \mathbb{E}_*^p(\Pi_\alpha) + \alpha \mathbb{E}_*^p(\Pi_{-\beta}) \leqslant \max\{\mathbb{E}_*^p(\Pi_\alpha), \mathbb{E}_*^p(\Pi_{-\beta})\}.$$

Applying subsequent basic systems to the one of Π_α and $\Pi_{-\beta}$ with larger $\mathbb{E}_*^p(\cdot)$, we conclude

$$\mathbb{E}_*^p(K) + c_3^p \varepsilon^2 \leqslant \mathbb{E}_*^p(\Pi) + c_3^p \varepsilon^2 \leqslant \mathbb{E}_*^p(T^2).$$

Turning to $\mathbb{E}_3^p(K)$, the major difference of the argument is that we need a third interval for the third vertex of the triangle. Writing $I_1 = [a, b]$, we define $\widetilde{I}_2 = I_2$, and

$$\widetilde{I}_0 = a + \tfrac{1}{10}(I_1 - a) \text{ and } \widetilde{I}_1 = b + \tfrac{1}{10}(I_1 - b).$$

In addition, we shorten $\sigma_t^*(y)$ for $y \in I_1$ to

$$\tilde{\sigma}_t(y) = \tfrac{1}{80}(\sigma_t(y) - m_t(y)) + m_t(y).$$

We change our main claim (3.38) to the following. There exists an absolute constant $c_4 > 0$, such that for any $y_0 \in \widetilde{I}_0$, $y_1 \in \widetilde{I}_1$ and $y_2 \in \widetilde{I}_2$, the integral

$$\tilde{f}(t) = \int_{\tilde{\sigma}_t(y_0)} \int_{\tilde{\sigma}_t(y_1)} \int_{\sigma_t(y_2)} A([z_0, z_1, z_2])^p \, dz_0 dz_1 dz_2$$

satisfies

$$\alpha \tilde{f}(-\beta) + \beta \tilde{f}(\alpha) \geqslant \tilde{f}(0) + c_4^p \varepsilon. \tag{3.39}$$

Now the proof of Lemma 3.16 can be completed along the argument above by introducing the obvious alterations. □

COROLLARY 3.17. *There exists a positive absolute constant $\tilde{c}$ such that if $p \geqslant 1$, and $A(K) = (1+\varepsilon)A(T_K)$ for a convex disc K, then*

$$\mathbb{E}_3^p(K) \leqslant (1 - \tilde{c}^p \varepsilon^2)\mathbb{E}_3^p(T^2) \quad \text{and} \quad \mathbb{E}_*^p(K) \leqslant (1 - \tilde{c}^p \varepsilon^2)\mathbb{E}_*^p(T^2).$$

PROOF. We present the argument only for $\mathbb{E}_3^p(K)$. Let $\hat{c}$ and ε_0 come from Lemma 3.16. We may assume that K is an m-gon for $m \geqslant 4$ by continuity, and that $A(K) > (1+\varepsilon_0)A(T_K)$ by Lemma 3.16. It follows by (3.30), that there exist $m-3$ consecutive basic systems that induce a continuous deformation of K into a triangle in a way such that $\mathbb{E}_3^p(\cdot)$ is strictly increasing during the deformation. Therefore there exists a polygon K' such that $\mathbb{E}_3^p(K') > \mathbb{E}_3^p(K)$, and $A(K') = (1+\varepsilon_0)A(T_{K'})$. Now we apply Lemma 3.16 to K', and using $\varepsilon < 3$, we deduce

$$\mathbb{E}_3^p(K) < \mathbb{E}_3^p(K') \leqslant (1 - \hat{c}^p \varepsilon_0^2)\mathbb{E}_3^p(T^2) < (1 - \tfrac{\hat{c}^p \varepsilon_0^2}{9} \cdot \varepsilon^2)\mathbb{E}_3^p(T^2). \qquad \square$$

Having Corollary 3.17, Theorem 3.7 is a consequence of the following.

LEMMA 3.18. *If $\delta_{BM}(K, T^2) = 1 + \delta$ for a convex disc K, then*

$$(1+\delta)A(T_K) \leqslant A(K) < (1+\delta)^2 A(T_K).$$

PROOF. The upper bound is consequence of the fact that by the definition of the Banach-Mazur distance, there exists a triangle $T' \subset K$, and $x \in T'$, such that $K \subset (1+\delta)(T'-x)+x$. For the lower bound, we may assume that T_K is a regular triangle of edge length one. Let p_1, p_2, p_3 be the vertices of T_K, and let q_1, q_2, q_3 be the such that p_i is the midpoint of $[q_j, q_k]$, $\{i,j,k\} = \{1,2,3\}$. If $\{i,j,k\} = \{1,2,3\}$, then let t_i

be the maximal distance of points of $K \cap [q_i, p_j, p_k]$ from $[p_j, p_k]$. On the one hand,

$$A(K) \geqslant A(T_K) + (t_1 + t_2 + t_3)/2 = (1 + \tfrac{2}{\sqrt{3}}(t_1 + t_2 + t_3))A(T_K).$$

On the other hand, K is contained in a regular triangle, that is similarly situated to T_K, and whose height is $\frac{\sqrt{3}}{2} + t_1 + t_2 + t_3$. It follows that

$$1 + \delta \leqslant (\tfrac{\sqrt{3}}{2} + t_1 + t_2 + t_3)/\tfrac{\sqrt{3}}{2} = 1 + \tfrac{2}{\sqrt{3}}(t_1 + t_2 + t_3) \leqslant A(K)/A(T_K). \qquad \square$$

That the exponent 2 in the error term δ^2 is optimal is shown by the example of the closure of $T \backslash \delta T$, where T is a triangle such that o is a vertex.

3.6 Stability of Petty projection inequality

Theorem 3.2 readily implies the stability version of the Busemann-Petty centroid inequality (3.5), using (3.4). Here we also derive the stability version of Petty's projection inequality (cf. [Lut93]). Given a convex body K, its *projection body* ΠK is defined by its support function

$$h_{\Pi K}(u) = V_{d-1}(p_u(K)).$$

The Petty projection inequality states that the quantity

$$V_d(K)^{d-1} V_d(\Pi^*(K))$$

is maximised for ellipsoids. Citing formula (5.7) of [Lut93] and using (3.4), we arrive to that if $V_d(K) = 1$, then

$$\frac{1}{V_d(K)^{d-1} V_d(\Pi^* K)} \geqslant \left(\frac{d+1}{2}\right)^d \frac{V_d(\Gamma(\Pi^* K))}{V_d(\Pi^* K)} = (d+1)^d \mathbb{E}_o^1(\Pi^* K). \tag{3.40}$$

Let $\delta_{BM}(K, B^d) = 1 + \delta$. Bourgain and Lindenstrauss [BL88] proved that there exists a constant C depending on d, so that

$$\delta_{BM}(\Pi K, B^d) \geqslant 1 + C\delta^{(d^2+5d)/2}.$$

Referring to $\delta_{BM}(\Pi K, B^d) = \delta_{BM}(\Pi^* K, B^d)$, Theorem 3.2 implies that there exists a constant c depending on d only, so that

$$\mathbb{E}_o^1(\Pi^* K) \geqslant (1 + C'\delta^{d(d+3)(d+5)/2})\mathbb{E}_o^1(B^d).$$

Thus, from (3.40) we obtain that

$$V_d(K)^{d-1}V_d(\Pi^* K) \leqslant (1 + c\delta^{d(d+3)(d+5)/2})^{-1}V_d(B^d)^{d-1}V_d(\Pi^* B^d).$$

We note that the stability version of the Busemann intersection inequality (3.3) would also follow by verifying a statement of the following type. If K is a convex body in $\mathbb{R}^d$, and $\delta_{BM}(K, B^d) = 1 + \delta$ for some $\delta > 0$, then there exist $\nu, \eta > 0$ (depending on δ) so that $\delta_{BM}(K \cap u^\perp, B^{d-1}) > 1 + \eta$ for a set of directions u of measure at least ν. The enthusiast would believe in such a statement with an absolute constant ν and $\eta = \delta^q$ for some $q > 0$.

Summary

The dissertation investigates three different problems, which are connected via the underlying, intuitive geometric motivation. The results are obtained by using geometric, combinatorial and analytic tools. We note that all the topics discussed here originate from the first half of the 20th century, hence they are well embedded in the research field of discrete and convex geometry. The dissertation is based on the following three publications of the author.

- G. Ambrus, A. Bezdek, F. Fodor, A Helly-type transversal theorem for n-dimensional unit balls, *Archiv der Mathematik* **86** (2006), no. 5, 470–480.
- G. Ambrus, F. Fodor, A new lower bound on the surface area of a Voronoi polyhedron, *Periodica Mathematica Hungarica* **53** (2006), no. 1-2., 45–58.
- G. Ambrus, K. J. Böröczky, Stability results for the volume of random simplices. Submitted to *American Journal of Mathematics*. pp. 1–26.

Transversals of unit balls

Chapter 1 deals with the following question. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. We say that a line ℓ is a *transversal* to $\mathcal{F}$, if it intersects every member of $\mathcal{F}$. If $\mathcal{F}$ has a transversal, then it is said to have *property* T. If every k or fewer members of $\mathcal{F}$ have a transversal, then $\mathcal{F}$ has *property* $T(k)$.

The question is the following: how can we guarantee that property T holds? In particular, we would like to derive the validity of T from $T(k)$ with some k. Such a setting is familiar from Helly's classical theorem, which states that if every at most $d+1$ members of a finite family of convex sets in $\mathbb{R}^d$ has a common point, then all the sets in the family intersect in a common point. Thus, such a transversal theorem can be understood as a generalisation of Helly's theorem.

It turns out that the above goal is too optimistic, if one considers all families of convex bodies: there exists no such general result. Even for families that consist of pairwise disjoint translates of an arbitrary convex body in $\mathbb{R}^3$, no such result exists, as was shown by Holmsen and Matoušek [HM04].

Our work considers the case when $\mathcal{F}$ consists of unit balls in $\mathbb{R}^d$. We are typically interested in large d's. The first related result by Hadwiger [Had56] states that for any family of *thinly distributed* balls in $\mathbb{R}^d$, the property $T(d^2)$ implies T, where a family of balls is thinly distributed if the distance between the centers of any two balls is at least twice the the sum of their radii. Prior to our result, in [HKL03] and [CGH05] it was proved that or any family of pairwise disjoint unit balls in $\mathbb{R}^3$, $T(11)$ implies T.

We impose a condition on the pairwise distances of the centres, which is weaker than Hadwiger's condition, but stronger than disjointness. This will be referred as the *distance condition.*

THEOREM 1.1. *Let $d \geqslant 2$, and $\mathcal{F}$ be a family of unit balls in $\mathbb{R}^d$ with the property that the mutual distances of the centres are at least $2\sqrt{2+\sqrt{2}}$. If every at most d^2 members of $\mathcal{F}$ have a common line transversal, then all members do.*

The methods used to prove Theorem 1.1 have been pushed further since the publication of [ABF06]. After a series of results, Cheong, Goaoc, Holmsen and Petitjean [CGHP08] proved that for any system of disjoint unit balls in $\mathbb{R}^d$, $T(4d-1)$ implies T.

The proof of Theorem 1.1 is based on the following statement. Let $B_1, \ldots, B_m$ be disjoint unit balls in $\mathbb{R}^d$. Consider the set of all directed lines intersecting $B_1, \ldots, B_m$ in this order, and denote the set of unit direction vectors of these lines by $\mathcal{K}(B_1, \ldots, B_m)$.

THEOREM 1.2. *Let $\mathcal{F}_d = \{B_1, \ldots, B_m\}$ be a family of unit balls satisfying the distance condition. Then $\mathcal{K}(B_1, \ldots, B_m)$ is strictly spherically convex.*

The crucial advantage of Theorem 1.2 is that it reduces the original problem to a 3-dimensional one, which can be attacked by standard analytical tools.

After establishing the convexity of the cone of transversal directions, in Section 1.3 we prove that if a family $\mathcal{F}_d$ of unit balls satisfying the distance condition has a transversal, then all the transversals of $\mathcal{F}_d$ intersect the unit balls in the same order (or its reverse). This ordering is called a *geometric permutation* of $\mathcal{F}_d$. Thus, the distance condition implies that there is at most one geometric permutation of $\mathcal{F}_d$.

Finally, in Section 1.4, we prove Theorem 1.1 by using the previous results and invoking the strong version of the Spherical Helly Theorem.

A new bound for the Strong Dodecahedral Conjecture

The contents of Chapter 2 are to give an improvement on the lower bound on the surface area of a Voronoi cell in a unit ball packing.

A family $\mathcal{B}$ of unit balls in $\mathbb{R}^3$ forms a *packing* if no two members of $\mathcal{B}$ have a common interior point. We are mostly interested in how dense a packing of unit balls may be, where the *density* of a packing is the proportion of the space covered by the balls. We define this as the limit of the proportion of the volume of the covered part of a ball, where the centre of the ball is fixed and its radius tends to infinity. According Kepler's Conjecture [Kep66], the packing density of unit balls in $\mathbb{R}^3$ is $\pi/\sqrt{18} \approx 0.74078\ldots$, which is attained by a lattice packing. Among lattice packings, this is indeed the best possible, as was shown by Gauss [Gau40]. The general result was proved recently by Hales [Hal05].

In a ball packing, the *Voronoi cell* of a ball $B \in \mathcal{B}$ is the set of points $x \in \mathbb{R}^3$ with the property that x is closer to the centre of B than to any other centre in $\mathcal{B}$. It is well known that Voronoi cells are convex polyhedra, and we may in fact assume that they are polytopes. The Dodecahedral Conjecture, formulated by L. Fejes Tóth [FT43] in 1943, states that the minimal volume of a Voronoi cell in a 3-dimensional unit ball packing is at least as large as the volume of a regular dodecahedron of inradius 1. This problem has been recently settled in the affirmative by Hales and McLaughlin [HM]. K. Bezdek [Bez00] phrased the following generalised version of the Dodecahedral Conjecture in 2000.

Conjecture 2.1 (Strong Dodecahedral Conjecture). *The minimum surface area of a Voronoi cell in a unit ball packing in $\mathbb{R}^3$ is at least as large as the surface area of the regular dodecahedron circumscribed about the unit ball, that is* $16.6508\ldots$.

In Chapter 2, we prove the following statement.

Theorem 2.2. *The surface area of a Voronoi cell in a unit ball packing in $\mathbb{R}^3$ is at least* $16.1977\ldots$.

This is currently the best estimate related to the problem. Prior to our result, the strongest bound was given by K. Bezdek and E. Daróczy-Kiss [BDK05], who, based on Muder's ideas ([Mud88] and [Mud93]), established the lower bound $16.1445\ldots$. Our improvement follows these lines as well.

In the proof, the cones suspended by the faces of the Voronoi cell are replaced with cones of special types in such a way, that the surface to solid angle ratio does not increase. The obtained configurations belong to a restricted class, in which the minimiser of the surface area is found by standard analytic methods.

In Section 2.2, the replacement steps are established. The cones used for replacements are the following. A *right circular cone* (RCC) is a cone whose base is a circular disk and its apex lies on the line perpendicular to the disk passing through its center. A *shaved circle* is the intersection of a disk and a convex polygon that contains the center of the disk. A *shaved right circular cone* (SRCC) is a cone whose base is a shaved circle and its apex lies on the line perpendicular to the disk and passing through its center. The desired replacements with RCC's or SRCC's are achieved via a series of basic replacement steps. Then, in Section 2.3, the surface to solid angle ratio of these special cones are further approximated.

Finally, in Section 2.4, the optimal configuration is determined using the previous approximations by a quite strenuous calculation. The minimal configuration has 13 identical faces and one face of a smaller solid angle. However, these faces cannot be joined to form a polytope, which accounts for the error between our estimate and the conjectured extremal value.

Stability results for the volume of random simplices

The following question serves as the motivation for Chapter 3. Given a convex body K in $\mathbb{R}^d$, what is the expected value of the volume of a random simplex in K? We work with two (or, rather, three) models: in the first, all the vertices of the simplex are chosen uniformly and independently from K, while in the second, one vertex is at a fixed position – in a special case, this is $\gamma(K)$, the centroid of K. We are interested in other moments as well, and also, we would like the answer to be invariant under affine transformations.

Definition 3.1. *Let K be a convex body in $\mathbb{R}^d$. For any $n \geqslant d+1$ and $p > 0$, let*

$$\mathbb{E}_n^p(K) = V(K)^{-n-p} \int_K \cdots \int_K V([x_1, \ldots, x_n])^p \, dx_1 \ldots dx_n.$$

Further, for a fixed $x \in \mathbb{R}^d$, *let*

$$\mathbb{E}_x^p(K) = V(K)^{-d-p} \int_K \dots \int_K V([x, x_1, \dots, x_d])^p \, dx_1 \dots dx_d.$$

Specifically, we write $\mathbb{E}_*^p(K)$ *for* $\mathbb{E}_x^p(K)$, *when* $x = \gamma(K)$.

These quantities have many connections to other concepts; for example, Sylvester's problem, the volume of centroid bodies and intersection bodies, the volume of Legendre's ellipsoid, Busemann's random simplex inequality, the Busemann-Petty centroid inequality, and so on. These links are elucidated in Section 3.1.

One is mostly interested in the the minimisers and maximisers of the above expectations among convex bodies. The search of these dates back to the early 20th century, see Blaschke ([Bla17] and [Bla23]). The minimisers are known in full generality.

THEOREM 3.2. (Blaschke [Bla23], Busemann [Bus53], Groemer[Gro74]) *For any convex body* K *in* $\mathbb{R}^d$, *for any* $p \geqslant 1$, *and for any* $n \geqslant d+1$, *we have*

$$\mathbb{E}_o^p(K) \geqslant \mathbb{E}_o^p(B^d) \quad and \quad \mathbb{E}_*^p(K) \geqslant \mathbb{E}_*^p(B^d) \quad and \quad \mathbb{E}_n^p(K) \geqslant \mathbb{E}_n^p(B^d).$$

Here $\mathbb{E}_o^p(K) = \mathbb{E}_o^p(B^d)$ *if and only if* K *is an o-symmetric ellipsoid, and* $\mathbb{E}_*^p(K) = \mathbb{E}_*^p(B^d)$ *or* $\mathbb{E}_n^p(K) = \mathbb{E}_n^p(B^d)$ *if and only if* K *is an ellipsoid.*

As for the maximisers, the Simplex conjecture states that for any convex body K in $\mathbb{R}^d$, and for any $p \geqslant 1$ and $n \geqslant d+1$, $\mathbb{E}_*^p(K) \leqslant \mathbb{E}_*^p(T^d)$ and $\mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(T^d)$, with equality if and only if K is a simplex. This is verified only in the plane.

THEOREM 3.4. ([Bla17],[DL91],[Gia92],[CCG99]) *If* $K \subset \mathbb{R}^2$ *is a convex disc, then for any* $n \geqslant 3$ *and* $p \geqslant 1$, $\mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(T^2)$ *and* $\mathbb{E}_*^p(K) \leqslant \mathbb{E}_*^p(T^2)$, *with equality if and only if* K *is a triangle.*

The importance of the Simplex conjecture stems from the fact that the affirmative answer to it would imply the Slicing conjecture.

In Chapter 3 of the dissertation, we provide the corresponding stability estimates for Theorems 3.2 and 3.4. The results are formulated with the use of the *Banach-Mazur distance* $\delta_{\mathrm{BM}}(K, M)$ of the convex bodies K and M, which is defined by $\delta_{\mathrm{BM}}(K, M) = \min\{\lambda \geqslant 1 : K - x \subset \Phi(M - y) \subset \lambda(K - x)\}$, where $\Phi \in \mathrm{GL}_d$ and $x, y \in \mathbb{R}^d$. Our results are as follows.

Theorem 3.6. *If K is a convex body in $\mathbb{R}^d$ with $\delta_{\mathrm{BM}}(K, B^d) = 1 + \delta$ for $\delta > 0$, then for any $p \geqslant 1$,*

$$\mathbb{E}_*^p(K) \geqslant (1 + \gamma^p \delta^{d+3}) \mathbb{E}_o^p(B^d)$$
$$\mathbb{E}_{d+1}^p(K) \geqslant (1 + \gamma^p \delta^{d+3}) \mathbb{E}_{d+1}^p(B^d),$$

where the constant $\gamma > 0$ depends on d only. Moreover, if K is centrally symmetric, then the error terms can be replaced by $\gamma^p \delta^{(d+3)/2}$.

Theorem 3.7. *If K is a planar convex body with $\delta_{\mathrm{BM}}(K, T^2) = 1 + \delta$ for some $\delta > 0$, and $p \geqslant 1$, then*

$$\begin{aligned} \mathbb{E}_*^p(K) &\leqslant (1 - c^p \delta^2) \mathbb{E}_*^p(T^2) \\ \mathbb{E}_3^p(K) &\leqslant (1 - c^p \delta^2) \mathbb{E}_3^p(T^2), \end{aligned}$$

where c is a positive absolute constant. This estimate is asymptotically sharp as δ tends to zero.

For the proof of Theorem 3.6, we first assume that K is a symmetric convex body in John's position, i.e. the unique ellipsoid of maximal volume inscribed in K is the unit ball. The core lemma estimates the change of the expectation when applying one step of Steiner symmetrisation in a suitable changed direction. The general result is then obtained by invoking a recent result of Böröczky [Bör], which estimates the Banach-Mazur distance between a convex body K and a symmetric convex body which is obtained by the limit of Steiner symmetrisations from K. We note that the bound of Theorem 3.6 is almost asymptotically sharp in terms of δ: there is an example, where the error is of order $\varepsilon^{(d+1)/2}$.

The stability version of the maximum inequality, Theorem 3.7 in the plane is obtained by the method of linear shadow systems, that were introduced by Campi, Colesanti and Gronchi [CCG99]. We assume that the triangle inscribed in K of maximal area is an equilateral triangle. With the aid of basic linear shadow systems, first we reduce the problem to polygons with at most 6 vertices. These polygons are then further modified in order to obtain the desired inequality.

To conclude the chapter, in Section 3.6 we derive the stability version of the Petty projection inequality from Theorem 3.2.

Összefoglalás

A disszertációban három problémát vizsgálunk, melyek főleg a megoldásukat inspiráló geometriai intuíción keresztül kapcsolódnak. Bizonyításaink geometriai, kombinatorikus és analitikus eszközöket használnak. Megjegyezzük, hogy a vizsgált területek gyökerei a 20. század első felébe nyúlnak vissza, s így a kutatott kérdések számos szállal kapcsolódnak a diszkrét és konvex geometria különböző területeihez.

Az értekezés az alábbi három publikáción alapszik:

- G. Ambrus, A. Bezdek, F. Fodor, A Helly-type transversal theorem for n-dimensional unit balls, *Archiv der Mathematik* **86** (2006), no. 5, 470–480.
- G. Ambrus, F. Fodor, A new lower bound on the surface area of a Voronoi polyhedron, *Periodica Mathematica Hungarica* **53** (2006), no. 1-2., 45–58.
- G. Ambrus, K. J. Böröczky, Stability results for the volume of random simplices. Publikálásra benyújtva, *American Journal of Mathematics.* pp. 1–26.

Egységgömbök transzverzálisai

Az 1. Fejezetben a következő kérdéskörrel foglalkozunk. Legyen $\mathcal{F}$ $\mathbb{R}^d$-beli halmazok egy rendszere. Az ℓ az $\mathcal{F}$ rendszer *transzverzálisa*, ha minden benne levő halmazt metsz. Az $\mathcal{F}$ rendszerre *teljesül a T tulajdonság*, ha van transzverzálisa, és *teljesül rá a $T(k)$ tulajdonság*, ha bármely legfeljebb k elemének van transzverzálisa (itt $k \geqslant 1$ egész szám).

Az alapkérdés a következő: hogyan tudjuk garantálni a T tulajdonság teljesülését? Speciálisan, szeretnénk belátni, hogy ha $\mathcal{F}$-re teljesül $T(k)$ (valamely k-ra), akkor van transzverzálisa is. Ez a felállás ismerős a klasszikus Helly-tételből, mely szerint ha az $\mathbb{R}^d$-beli konvex halmazok egy véges rendszerének bármely legfeljebb $d+1$ tagja metsző, akkor a rendszer összes tagjának van közös pontja. Tehát a fenti típusú transzverzális eredmények a "0-dimenziós" Helly-tétel "1-dimenziós" általánosításának is tekinthetők.

Az, hogy minden további megszorítás nélkül bizonyítsunk a fenti sémának megfelelő transzverzális eredményt, túl optimista cél. Holmsen és Matoušek eredménye [HM04] mutatja, hogy még olyan tétel sem adható, mely az összes olyan rendszerre igaz, ami egy R^3-beli konvex test páronként diszjunkt eltoltjaiból áll.

Az általunk vizsgált szituációban $\mathcal{F}$ az $\mathbb{R}^d$-beli egységgömbök egy rendszere, ahol d tetszőleges pozitív egész. Az első kapcsolódó eredmény Hadwigerhez köthető [Had56], aki belátta, hogy $T(d^2)$-ből következik T bármely $\mathbb{R}^d$-beli gömbök *ritkán elosztott* rendszerére: itt bármely két gömb középpontjának távolsága legalább 2-szer akkora, mint sugaraik összege. Egy másik vonatkozó eredmény szerint, ld. [HKL03] és [CGH05], $T(11)$-ből következik T tetszőleges $\mathbb{R}^3$-beli diszjunkt egységgömbökből álló rendszerre.

Az általunk használt feltétel erősebb, mint a diszjunktság, de gyengébb a Hadwiger-féle kritériumánál; a továbbiakban csak *távolságfeltételként* fogunk rá hivatkozni.

1.1. TÉTEL. *Legyen $d \geqslant 2$, és $\mathcal{F}$ $\mathbb{R}^d$-beli egységgömbök egy rendszere, melyekre teljesül, hogy bármely kettő középpontjának a távolsága legalább $2\sqrt{2+\sqrt{2}}$. Ha $\mathcal{F}$ bármely legfeljebb d^2 elemének létezik közös transzverzálisa, akkor az összes gömbnek is létezik transzverzális egyenese.*

Az [ABF06] cikk publikálása óta tovább folyt a kutatás a témában. Ennek eredményeképp, Cheong, Goaoc, Holmsen és Petitjean [CGHP08] az itt alkalmazottakhoz hasonló módszerekkel bebizonyította, hogy tetszőleges, diszjunkt, $\mathbb{R}^d$-beli egységgömbök rendszerére $T(4d-1)$ implikálja T-t.

Az 1.1. Tétel bizonyítása a következő állításon alapszik. Legyenek $B_1, \dots, B_m$ diszjunkt $\mathbb{R}^d$-beli egységgömbök. Vegyük azon irányított egyeneseket, melyek a $B_1, \dots, B_m$ gömböket az indexüknek megfelelő sorrendben metszik, és jelölje $\mathcal{K}(B_1, \dots, B_m)$ ezen egyenesek (egység hosszú) irányvektorainak halmazát.

1.2. TÉTEL. *Legyen $\mathcal{F}_d = \{B_1, \dots, B_m\}$ egységgömbök egy olyan rendszere, mely teljesíti a távolságfeltételt. Ekkor $\mathcal{K}(B_1, \dots, B_m)$ gömbi konvex halmaz.*

Így a problémát egy 3-dimenziós kérdésre redukáljuk, amely hatékonyan kezelhető analitikus módszerekkel. Ezután bebizonyítjuk, hogy ha az egységgömbökből álló $\mathcal{F}_d$ rendszer teljesíti a távolságfeltételt valamint létezik transzverzálisa, akkor bármely transzverzálisa ugyanabban a sorrendben (vagy a fordítottjában) metszi a gömböket. Ilyen indukált rendezést az $\mathcal{F}_d$ rendszer egy *geometriai permutációjának* nevezünk. Tehát a távolságfeltételből következik, hogy $\mathcal{F}_d$-nek legfeljebb egy geometriai permutációja létezik. Végül, az 1.1. Tételt a fenti eredmények és a gömbi Helly-tétel segítségével igazoljuk.

Alsó korlát az Erős Dodekahedrális Sejtésre

A 2. Fejezetben az egységgömb-pakolások Voronoi celláinak minimális felszínmértékére vonatkozó alsó korlátot javítjuk.

Az $\mathbb{R}^3$-beli egységgömbök egy $\mathcal{B}$ rendszerét *pakolásnak* nevezzük, ha semelyik két gömbnek nincs közös belső pontja. A legfontosabb kapcsolódó kérdés, hogy mennyire lehet *sűrű* egy gömbpakolás, ahol a sűrűség alatt a lefedett tér arányát érjtük: egy rögzített középpontú gömb sugarát a végtelenbe tartatva, a kapott arányok határértékeként definiáljuk (már ha ez a határérték létezik). Kepler [Kep66] klasszikus sejtése szerint, a 3-dimenziós egységgömbök pakolási sűrűsége $\pi/\sqrt{18} \approx 0.74078\ldots$, amelyet egy rácsszerű elrendezéssel érhetünk el. A rácsszerű pakolásokra szorítkozva, ez a korlát valóban optimális, ahogy Gauss megmutatta [Gau40]. A Kepler-sejtést Hales igazolta [Hal05].

Tekintsünk egy $\mathcal{B}$ gömbpakolást. Egy B gömb Voronoi cellája azon pontok halmaza, melyek közelebb vannak B középpontjához, mint bármelyik másik gömbközépponthoz. Közismert, hogy a Voronoi cellák konvex poliéderek, és esetünkben feltehető, hogy politópok. A Dodekahedrális Sejtés állítása szerint, melyet Fejes Tóth László fogalmazott meg [FT43], egy egységgömb-pakolás tetszőleges Voronoi cellájának térfogata legalább akkora, mint az egységgömb köré írt szabályos dodekaéder térfogata. Ezt a közelmúltban Hales és McLaughlin igazolták [HM].

Bezdek Károly [Bez00] a Dodekahedrális Sejtést a következőképpen általánosította.

2.1. Sejtés. (Erős Dodekahedrális Sejtés). *Egy $\mathbb{R}^3$-beli egységgömbpakolás tetszőleges Voroni cellájának felszíne legalább akkora, mint az egységgömb köré írt szabályos dodekaéder felszíne:* $16.6508\ldots$.

Mi a következő korlátot adjuk.

2.2 Tétel. *Egy $\mathbb{R}^3$-beli egységgömbpakolás tetszőleges Voroni cellájának felszíne legalább* $16.1977\ldots$.

Jelenleg ez a problémára vonatkozó legjobb alsó korlát. Korábban Bezdek K. és Daróczy-Kiss E. [BDK05] adott alsó becslést D. Muder [Mud88],[Mud93] gondolatmenetének felhasználásával. Mi is ezt az utat követjük.

A bizonyítás alapötlete a következő. A Voronoi cella lapjai által kifeszített kúpokat (a továbbiakban lapkúpokat) speciális kúpokkal helyettesítjük oly módon, hogy a felület és a térszög aránya nem növekszik. Az így kapott, szűkebb osztályba tartozó konfigurációk közötti optimumot analitikus eszközökkel határozzuk meg.

A 2.2. alfejezetben részletezzük a helyettesítési eljárást. A használt speciális lapkúpok a következőek. A *merőleges körkúp* (RCC) alapja egy körlap, csúcsa pedig az ennek középpontján áthaladó, a kör síkjára merőleges egyenesen helyezkedik el. *Metszett körnek* hívjuk egy körlemez és egy ennek a középpontját tartalmazó szabályos sokszög metszetét. A *merőleges metszett körkúp* (SRCC) pedig olyan kúp, melynek alapja egy metszett kör, csúcsa pedig ismét a kör középpontja fölött helyezkedik el. Az eljárásban elemi helyettesítések sorozatával a lapkúpokat RCC illetve SRCC típusú kúpokkal helyettesítjük. Ezeknek a speciális kúpoknak a felszín-térszög arányát további, egyszerűbben kezelhető függvényekkel approximáljuk.

Végül, a 2.4. alfejezetben a korábbi becslések, valamint egy technikai számolás segítségével meghatározzuk az optimális konfigurációt. Ez 13 egybevágó, valamint egy kisebb térszögű lapból áll. Ezek a lapok azonban nem illeszthetőek össze egy politóppá; ez okozza a becslésünk és a sejtett érték közötti eltérést.

Véletlen szimplexek térfogatára vonatkozó egyenlőtlenségek stabilitása

A 3. Fejezet motivációjaként a következő kérdés szolgál. Legyen K egy $\mathbb{R}^d$-beli konvex test. Mi a várható értéke egy K-beli véletlen szimplex térfogatának? Három modellt vizsgálunk: az elsőnél, a szimplex csúcsait függetlenül, egyenletes eloszlással választjuk K-ból; a második modellnél, egy csúcs rögzített helyzetben van; míg a harmadiknál, a rögzített csúcs a K súlypontja, $\gamma(K)$. A várható érték mellett más momentumokat is vizsgálunk, és a mennyiségeket affin invariáns módon mérjük.

3.1. Definíció. *Legyen $K \subset \mathbb{R}^d$ konvex test. Tetszőleges $n \geqslant d+1$ és $p > 0$ esetén, vezessük be a következő jelölést:*

$$\mathbb{E}_n^p(K) = V(K)^{-n-p} \int_K \cdots \int_K V([x_1, \ldots, x_n])^p \, dx_1 \ldots dx_n.$$

Továbbá, valamely rögzített $x \in \mathbb{R}^d$-re, legyen

$$\mathbb{E}_x^p(K) = V(K)^{-d-p} \int_K \cdots \int_K V([x, x_1, \ldots, x_d])^p \, dx_1 \ldots dx_d.$$

Abban a speciális esetben, amikor $x = \gamma(K)$, $\mathbb{E}_x^p(K)$ helyett $\mathbb{E}_*^p(K)$-t írunk.

A fent bevezetett mennyiségek számos kapcsolattal rendelkeznek; ezek közé tartozik Sylvester kérdése, a centroid test és a metszési test térfogata, a Legendre-ellipszoid térfogata, a Busemann véletlen szimplex egyenlőtlenség, a Busemann-Petty centroid egyenlőtlenség, s í.t. Ezeket az összefüggéseket a 3.1. alfejezetben tárgyaljuk.

A legérdekesebb kérdés az, hogy mely K konvex test esetén vétetik fel a fenti mennyiségek minimuma ill. maximuma. Ez a probléma a 20. század elejéről származik, ld. Blaschke [Bla17], [Bla23]. A minimumok esete teljesen megoldott.

3.2. Tétel. (Blaschke [Bla23], Busemann [Bus53], Groemer[Gro74]) *Tetszőleges $K \subset \mathbb{R}^d$ konvex test, $p \geqslant 1$, és $n \geqslant d+1$ esetén,*

$$\mathbb{E}_o^p(K) \geqslant \mathbb{E}_o^p(B^d) \quad , \quad \mathbb{E}_*^p(K) \geqslant \mathbb{E}_*^p(B^d) \quad , \text{ és } \ \mathbb{E}_n^p(K) \geqslant \mathbb{E}_n^p(B^d).$$

Továbbá, $\mathbb{E}_o^p(K) = \mathbb{E}_o^p(B^d)$ teljesül pontosan akkor, ha K egy o-szimmetrikus ellipszoid, valamint $\mathbb{E}_*^p(K) = \mathbb{E}_*^p(B^d)$ illetve $\mathbb{E}_n^p(K) = \mathbb{E}_n^p(B^d)$ pontosan akkor, ha K ellipszoid.

A maximumok esete sokkal kevésbé ismert. A Szimplex Sejtés szerint tetszőleges $K \subset \mathbb{R}^d$ konvex test, $p \geqslant 1$ és $n \geqslant d+1$ esetén, $\mathbb{E}_*^p(K) \leqslant \mathbb{E}_*^p(T^d)$ és $\mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(T^d)$, ahol egyenlőség pontosan akkor áll, ha K szimplex. Ez csak a síkon bizonyított.

3.4. Tétel. ([Bla17],[DL91],[Gia92],[CCG99]) *Tetszőleges $K \subset \mathbb{R}^2$ konvex lemez, $n \geqslant 3$ és $p \geqslant 1$ esetén, $\mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(T^2)$ és $\mathbb{E}_*^p(K) \leqslant \mathbb{E}_*^p(T^2)$. Egyenlőség pontosan akkor áll fenn, ha K háromszög.*

A Szimplex Sejtés fontossága onnan ered, hogy következne belőle a magas dimenziós konvex geometria egyik központi sejtése, a Hipersík Sejtés.

A disszertáció 3. fejezetében a 3.2. és 3.4. Tételek stabilitási változatait bizonyítjuk. Eredményeinket a *Banach-Mazur távolság* segítségével fogalmazzuk meg: a K és M konvex testek Banach-Mazur távolsága $\delta_{\mathrm{BM}}(K, M) = \min\{\lambda \geqslant 1 : K - x \subset \Phi(M - y) \subset \lambda(K - x)\}$, ahol Φ a GL_d-n, míg x, y az $\mathbb{R}^d$-n fut végig. Eredményeink a következőek.

3.6. Tétel. *Ha a $K \subset \mathbb{R}^d$ konvex testre $\delta_{\mathrm{BM}}(K, B^d) = 1 + \delta$ valamely $\delta > 0$-val, akkor tetszőleges $p \geqslant 1$ esetén,*

$$\mathbb{E}^p_*(K) \geqslant (1 + \gamma^p \delta^{d+3}) \mathbb{E}^p_o(B^d)$$
$$\mathbb{E}^p_{d+1}(K) \geqslant (1 + \gamma^p \delta^{d+3}) \mathbb{E}^p_{d+1}(B^d),$$

ahol a $\gamma > 0$ konstans egyedül d-től függ. Továbbá, ha K centrálszimmetrikus, akkor a hibatag $\gamma^p \delta^{(d+3)/2}$-ra cserélhető.

3.7. Tétel. *Legyen K konvex lemez, melyre $\delta_{\mathrm{BM}}(K, T^2) = 1 + \delta$ valamely $\delta > 0$-val. Ekkor tetszőleges $p \geqslant 1$-re,*

$$\mathbb{E}^p_*(K) \leqslant (1 - c^p \delta^2) \mathbb{E}^p_*(T^2)$$
$$\mathbb{E}^p_3(K) \leqslant (1 - c^p \delta^2) \mathbb{E}^p_3(T^2),$$

ahol c egy pozitív abszolút konstans. A becslések aszimptotikusan élesek, ha $\delta \to 0$.

A 3.6. Tétel bizonyításánál először feltesszük, hogy K centrálszimmetrikus konvex test, amely John pozícióban van, azaz a bele írható legnagyobb térfogatú ellipszoid az egységgömb. Fő lemmánk segítségével azt becsüljük, hogy mennyit változnak a kérdéses mennyiségek egy megfelelően választott Steiner szimmetrizáció elvégzésekor. Az általános eredményt ezután Böröczky Károly [Bör] egy tételét felhasználva kapjuk, mely becslést ad egy K konvex test, és a belőle Steiner szimmetrizáltak határértékeként kapott szimmetrikus test Banach-Mazur távolságára. Megjegyezzük, hogy a 3.6. Tétel becslése aszimptotikusan közel optimális: a 3.4. alfejezetben közölt példa esetén a hibatag $\varepsilon^{(d+1)/2}$ nagyságrendű.

A 3.7. Tételt a lineáris árnyék-rendszerek technikájának segítségével bizonyítjuk, melyet Campi, Colesanti és Gronchi [CCG99] vezetett be. Ez a "shaking" technika egyfajta általánosítása. Feltesszük, hogy a K-ba írható maximális területű háromszög szabályos, majd a problémát legfeljebb 6 csúcsú poligonok esetére redukáljuk. A fő nehézséget ezeknek a poligonoknak a további módosítása jelenti, amelyhez elemi árnyék-rendszereket használunk. A kívánt becsléshez egy technikai jellegű számolással jutunk.

A fejezetet a Petty vetítési egyenlőtlenség stabilitási változatának bizonyításával zárjuk.

References

[AB] G. Ambrus and K. J. Böröczky. Stability results for the volume of random simplices. *Submitted to the American Journal of Mathematics.*

[ABF06] G. Ambrus, A. Bezdek, and F. Fodor. A Helly-type transversal theorem for n-dimensional unit balls. *Arch. Math. (Basel)*, 86(5):470–480, 2006.

[AF06] G. Ambrus and F. Fodor. A new lower bound on the surface area of a Voronoi polyhedron. *Period. Math. Hungar.*, 53(1-2):45–58, 2006.

[Bal88] Keith Ball. Logarithmically concave functions and sections of convex sets in $\mathbf{R}^n$. *Studia Math.*, 88(1):69–84, 1988.

[Bal97] Keith Ball. An elementary introduction to modern convex geometry. In *Flavors of geometry*, volume 31 of *Math. Sci. Res. Inst. Publ.*, pages 1–58. Cambridge Univ. Press, Cambridge, 1997.

[Bár08] Imre Bárány. Random points and lattice points in convex bodies. *Bull. Amer. Math. Soc. (N.S.)*, 45(3):339–365, 2008.

[BB93] Imre Bárány and Christian Buchta. Random polytopes in a convex polytope, independence of shape, and concentration of vertices. *Math. Ann.*, 297(3):467–497, 1993.

[BB01] T. Bisztriczky and K. Böröczky, Jr. About the centroid body and the ellipsoid of inertia. *Mathematika*, 48(1-2):1–13 (2003), 2001.

[BBCH06] K. Bezdek, T. Bisztriczky, B. Csikós, and A. Heppes. On the transversal Helly numbers of disjoint and overlapping disks. *Arch. Math. (Basel)*, 87(1):86–96, 2006.

[BDK05] Károly Bezdek and Endre Daróczy-Kiss. Finding the best face on a Voronoi polyhedron—the strong dodecahedral conjecture revisited. *Monatsh. Math.*, 145(3):191–206, 2005.

[Bez00] Károly Bezdek. On a stronger form of Rogers's lemma and the minimum surface area of Voronoi cells in unit ball packings. *J. Reine Angew. Math.*, 518:131–143, 2000.

[BF87] T. Bonnesen and W. Fenchel. *Theory of convex bodies.* BCS Associates, Moscow, ID, 1987. Translated from the German and edited by L. Boron, C. Christenson and B. Smith.

[BL88] J. Bourgain and J. Lindenstrauss. Projection bodies. In *Geometric aspects of functional analysis (1986/87)*, volume 1317 of *Lecture Notes in Math.*, pages 250–270. Springer, Berlin, 1988.

[Bla17] W. Blaschke. Über affine Geometrie IX: Verschiedene Bemerkungen und Aufgaben. *Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.–Phys. Kl.*, 69:412–420, 1917.

[Bla18] W. Blaschke. Über affine Geometrie XI: eine minimum Aufgabe fr Legendres trägheits Ellipsod. *Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.–Phys. Kl.*, 70:72–75, 1918.

[Bla23] W. Blaschke. *Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie*. Springer, Berlin, 1923.

[BMMP88] J. Bourgain, M. Meyer, V. Milman, and A. Pajor. On a geometric inequality. In *Geometric aspects of functional analysis (1986/87)*, volume 1317 of *Lecture Notes in Math.*, pages 271–282. Springer, Berlin, 1988.

[Bör] K.J. Böröczky. Stability of the Blaschke-Santaló and the affine isoperimetric inequality. *Adv. Math.* Accepted.

[Bou86] J. Bourgain. On high-dimensional maximal functions associated to convex bodies. *Amer. J. Math.*, 108(6):1467–1476, 1986.

[BR01] Christian Buchta and Matthias Reitzner. The convex hull of random points in a tetrahedron: solution of Blaschke's problem and more general results. *J. Reine Angew. Math.*, 536:1–29, 2001.

[Buc84] Christian Buchta. Zufallspolygone in konvexen Vielecken. *J. Reine Angew. Math.*, 347:212–220, 1984.

[Bus53] Herbert Busemann. Volume in terms of concurrent cross-sections. *Pacific J. Math.*, 3:1–12, 1953.

[CCG99] Stefano Campi, Andrea Colesanti, and Paolo Gronchi. A note on Sylvester's problem for random polytopes in a convex body. *Rend. Istit. Mat. Univ. Trieste*, 31(1-2):79–94, 1999.

[CG06] Stefano Campi and Paolo Gronchi. Volume inequalities for L_p-zonotopes. *Mathematika*, 53(1):71–80 (2007), 2006.

[CGH05] Otfried Cheong, Xavier Goaoc, and Andreas Holmsen. Hadwiger and Helly-type theorems for disjoint unit spheres in $\mathbb{R}^3$. In *Computational Geometry (SCG'05)*, pages 10–15. ACM, New York, 2005.

[CGHP08] Otfried Cheong, Xavier Goaoc, Andreas Holmsen, and Sylvain Petitjean. Helly-type theorems for line transversals to disjoint unit balls. *Discrete Comput. Geom.*, 39(1-3):194–212, 2008.

[CGN05] Otfried Cheong, Xavier Goaoc, and Hyeon-Suk Na. Geometric permutations of disjoint unit spheres. *Comput. Geom.*, 30(3):253–270, 2005.

[Dan57] L. Danzer. Über ein Problem aus der kombinatorischen Geometrie. *Arch. Math. (Basel)*, 8:347–351, 1957.

[Deb70] H. E. Debrunner. Helly type theorems derived from basic singular homology. *Amer. Math. Monthly*, 77:375–380, 1970.

[DGK63] Ludwig Danzer, Branko Grünbaum, and Victor Klee. Helly's theorem and its relatives. In *Proc. Sympos. Pure Math., Vol. VII*, pages 101–180. Amer. Math. Soc., Providence, R.I., 1963.

[DL91] L. Dalla and D. G. Larman. Volumes of a random polytope in a convex set. In *Applied geometry and discrete mathematics*, volume 4 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 175–180. Amer. Math. Soc., Providence, RI, 1991.

[Eck93] Jürgen Eckhoff. Helly, Radon, and Carathéodory type theorems. In *Handbook of convex geometry, Vol. A, B*, pages 389–448. North-Holland, Amsterdam, 1993.

[FGP07] B. Fleury, O. Guédon, and G. Paouris. A stability result for mean width of L_p-centroid bodies. *Adv. Math.*, 214(2):865–877, 2007.

[FT43] L. Fejes Tóth. Über die dichteste Kugellagerung. *Math. Z.*, 48:676–684, 1943.

[Gar06] Richard J. Gardner. *Geometric tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2006.

[Gau40] C. F. Gauss. Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seber. *Göttingischen gelehrte Anzeigen, 9. Juli 1831. Also: J. Reine Angew. Math.*, 20:312–320, 1840.

[Gia] A. Giannopoulos. *Notes on Isotropic Convex Bodies.* Preprint. http://users.uoa.gr/∼apgiannop/.

[Gia92] Apostolos A. Giannopoulos. On the mean value of the area of a random polygon in a plane convex body. *Mathematika*, 39(2):279–290, 1992.

[GM04] A. A. Giannopoulos and V. D. Milman. Asymptotic convex geometry: short overview. In *Different faces of geometry*, volume 3 of *Int. Math. Ser. (N. Y.)*, pages 87–162. Kluwer/Plenum, New York, 2004.

[Goa09] Xavier Goaoc. Some discrete properties of the space of line transversals to disjoint balls. In *Non-linear Computational Geometry*, volume 151 of *IMA Volume Series*, pages 51–84. 2009.

[GPW91] I. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In J. Pach, editor, *New trends in Discrete and Computational Geometry*, pages 163–198. Springer-Verlag, 1991.

[Gro73] H. Groemer. On some mean values associated with a randomly selected simplex in a convex set. *Pacific J. Math.*, 45:525–533, 1973.

[Gro74] H. Groemer. On the mean value of the volume of a random polytope in a convex set. *Arch. Math. (Basel)*, 25:86–90, 1974.

[Gro94] H. Groemer. Stability theorems for ellipsoids and spheres. *J. London Math. Soc. (2)*, 49(2):357–370, 1994.

[Grü58] B. Grünbaum. On common transversals. *Arch. Math.*, 9:465–469, 1958.

[Grü60] B. Grünbaum. Common transversals for families of sets. *J. London Math. Soc.*, 35:408–416, 1960.

[Gru07] Peter M. Gruber. *Convex and discrete geometry*, volume 336 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Berlin, 2007.

[Had56] H. Hadwiger. Problem 107. *Nieuw Arch. Wiskunde*, 3(4):57, 1956. Solution: Wiskundige Opgaven 20 (1957), 27–29.

[Had57] H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie.* Springer-Verlag, Berlin, 1957.

[Hal05] Thomas C. Hales. A proof of the Kepler conjecture. *Ann. of Math. (2)*, 162(3):1065–1185, 2005.

[HKL03] Andreas Holmsen, Meir Katchalski, and Ted Lewis. A Helly-type theorem for line transversals to disjoint unit balls. *Discrete Comput. Geom.*, 29(4):595–602, 2003.

[HM] T. C. Hales and S. McLaughlin. A proof of the dodecahedral conjecture. *http://arXive:math.MG/9811079.*

[HM04] Andreas Holmsen and Jiří Matoušek. No Helly theorem for stabbing translates by lines in $\mathbb{R}^3$. *Discrete Comput. Geom.*, 31(3):405–410, 2004.

[Hol08] Andreas F. Holmsen. Recent progress on line transversals to families of translated ovals. In *Surveys on discrete and computational geometry*, volume 453 of *Contemp. Math.*, pages 283–297. Amer. Math. Soc., Providence, RI, 2008.

[Joh48] Fritz John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, pages 187–204. Interscience Publishers, Inc., New York, N. Y., 1948.

[Kep66] J. Kepler. *Strena seu de nive sexangula.* Tampach, Frankfurt, 1611. In English: *The Six-Cornered Snowflake*, Oxford, 1966.

[Kle69] Victor Klee. Research Problems: What is the Expected Volume of a Simplex Whose Vertices are Chosen at Random from a Given Convex Body? *Amer. Math. Monthly*, 76(3):286–288, 1969.

[Lin86] J. H. Lindsey, II. Sphere packing in $\mathbf{R}^3$. *Mathematika*, 33(1):137–147, 1986.

[Lut93] Erwin Lutwak. Selected affine isoperimetric inequalities. In *Handbook of convex geometry, Vol. A, B*, pages 151–176. North-Holland, Amsterdam, 1993.

[LZ91] Jeffrey C. Lagarias and Günter M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canad. J. Math.*, 43(5):1022–1035, 1991.

[MP89] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 64–104. Springer, Berlin, 1989.

[MR06] Mathieu Meyer and Shlomo Reisner. Shadow systems and volumes of polar convex bodies. *Mathematika*, 53(1):129–148 (2007), 2006.

[Mud88] Douglas J. Muder. Putting the best face on a Voronoĭ polyhedron. *Proc. London Math. Soc. (3)*, 56(2):329–348, 1988.

[Mud93] Douglas J. Muder. A new bound on the local density of sphere packings. *Discrete Comput. Geom.*, 10(4):351–375, 1993.

[Pet61] C. M. Petty. Centroid surfaces. *Pacific J. Math.*, 11:1535–1547, 1961.

[Ree74] W. J. Reed. Random points in a simplex. *Pacific J. Math.*, 54(2):183–198, 1974.

[Rog58] C. A. Rogers. The packing of equal spheres. *Proc. London Math. Soc. (3)*, 8:609–620, 1958.

[RS58] C. A. Rogers and G. C. Shephard. Some extremal problems for convex bodies. *Mathematika*, 5:93–102, 1958.

[Sch93] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[She64] G. C. Shephard. Shadow systems of convex sets. *Israel J. Math.*, 2:229–236, 1964.

[SW08] Rolf Schneider and Wolfgang Weil. *Stochastic and integral geometry*. Probability and its Applications (New York). Springer-Verlag, Berlin, 2008.

[Syl64] J. J. Sylvester. Problem 1491. *The Educational Times*, April 1864.

[Tve89] Helge Tverberg. Proof of Grünbaum's conjecture on common transversals for translates. *Discrete Comput. Geom.*, 4(3):191–203, 1989.

[Wen99] Rephael Wenger. Progress in geometric transversal theory. In *Advances in discrete and computational geometry (South Hadley, MA, 1996)*, volume 223 of *Contemp. Math.*, pages 375–393. Amer. Math. Soc., Providence, RI, 1999.