Theorems from the interface of convex geometry and analysis

Outline of Ph.D. dissertation

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1. Introduction

The dissertation investigates three different problems, which are connected via the underlying, intuitive geometric motivation. The results are obtained by using geometric, combinatorial and analytic tools. We note that all the topics discussed here originate from the first half of the 20th century, hence they are well embedded in the research field of discrete and convex geometry.

The dissertation is based on the following three publications.

- G. Ambrus, A. Bezdek, F. Fodor, A Helly-type transversal theorem for n-dimensional unit balls, *Archiv der Mathematik* **86** (2006), no. 5, 470–480.
- G. Ambrus, F. Fodor, A new lower bound on the surface area of a Voronoi polyhedron, *Periodica Mathematica Hungarica* **53** (2006), no. 1-2., 45–58.
- G. Ambrus, K. J. Böröczky, Stability results for the volume of random simplices. Submitted for publication, *American Journal of Mathematics*. pp. 1–26.

2. Transversals of unit balls

Chapter 1 deals with the following question. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. We say that a line ℓ is a *transversal* to $\mathcal{F}$, if it intersects every member of $\mathcal{F}$. If $\mathcal{F}$ has a transversal, then it is said to have *property* T. If every k or fewer members of $\mathcal{F}$ have a transversal, then $\mathcal{F}$ has *property* $T(k)$.

The question is the following: how can we guarantee that property T holds? In particular, we would like to derive the validity of T from $T(k)$ with some k. Such a setting is familiar from Helly's classical theorem, which states that if every at most $d+1$ members of a finite family of convex sets in $\mathbb{R}^d$ has a common point, then all the sets in the family intersect in a common point. Thus, such a transversal theorem can be understood as a generalisation of Helly's theorem.

It turns out that the above goal is too optimistic, if one considers all families of convex bodies: there exists no such general result. Even for families that consist of pairwise disjoint translates of an arbitrary convex body in $\mathbb{R}^3$, no such result exists, as was shown by Holmsen and Matoušek [HM04].

Our work considers the case when $\mathcal{F}$ consists of unit balls in $\mathbb{R}^d$. We are typically interested in large d's. The first related result by Hadwiger [Had56] states that for any family of *thinly distributed* balls in $\mathbb{R}^d$, the property $T(d^2)$ implies T, where a family of balls is thinly distributed if the distance between the centers of any two balls is at least twice the the sum of their radii. Prior to our result, in [HKL03] and [CGH05] it was proved that or any family of pairwise disjoint unit balls in $\mathbb{R}^3$, $T(11)$ implies T.

We impose a condition on the pairwise distances of the centres, which is weaker than Hadwiger's condition, but stronger than disjointness. This will be referred as the *distance condition.*

Theorem 1. *Let $d \geqslant 2$, and $\mathcal{F}$ be a family of unit balls in $\mathbb{R}^d$ with the property that the mutual distances of the centres are at least $2\sqrt{2+\sqrt{2}}$. If every at most d^2 members of $\mathcal{F}$ have a common line transversal, then all members do.*

The methods used to prove Theorem 1 have been pushed further since the publication of [ABF06]. After a series of results, Cheong, Goaoc, Holmsen and Petitjean [CGHP08] proved that for any system of disjoint unit balls in $\mathbb{R}^d$, $T(4d-1)$ implies T.

The proof of Theorem 1 is based on the following statement. Let $B_1, \ldots, B_m$ be disjoint unit balls in $\mathbb{R}^d$. Consider the set of all directed lines intersecting $B_1, \ldots, B_m$ in this order, and denote the set of unit direction vectors of these lines by $\mathcal{K}(B_1, \ldots, B_m)$.

THEOREM 2. *Let $\mathcal{F}_d = \{B_1, \ldots, B_m\}$ be a family of unit balls satisfying the distance condition. Then $\mathcal{K}(B_1, \ldots, B_m)$ is strictly spherically convex.*

The crucial advantage of Theorem 2 is that it reduces the original problem to a 3-dimensional one, which can be attacked by standard analytical tools.

After establishing the convexity of the cone of transversal directions, in Section 1.3 we prove that if a family $\mathcal{F}_d$ of unit balls satisfying the distance condition has a transversal, then all the transversals of $\mathcal{F}_d$ intersect the unit balls in the same order (or its reverse). This ordering is called a *geometric permutation* of $\mathcal{F}_d$. Thus, the distance condition implies that there is at most one geometric permutation of $\mathcal{F}_d$.

Finally, in Section 1.4, we prove Theorem 1 by using the previous results and invoking the strong version of the Spherical Helly Theorem.

3. A NEW LOWER BOUND FOR THE STRONG DODECAHEDRAL CONJECTURE

The contents of Chapter 2 are to give an improvement on the lower bound on the surface area of a Voronoi cell in a unit ball packing.

A family $\mathcal{B}$ of unit balls in $\mathbb{R}^3$ forms a *packing* if no two members of $\mathcal{B}$ have a common interior point. We are mostly interested in how dense a packing of unit balls may be, where the *density* of a packing is the proportion of the space covered by the balls. We define this as the limit of the proportion of the volume of the covered part of a ball, where the centre of the ball is fixed and its radius tends to infinity. According Kepler's Conjecture [Kep66], the packing density of unit balls in $\mathbb{R}^3$ is $\pi/\sqrt{18} \approx 0.74078\ldots$, which is attained by a lattice packing. This result was proved recently by Hales [Hal05].

In a ball packing, the *Voronoi cell* of a ball $B \in \mathcal{B}$ is the set of points $x \in \mathbb{R}^3$ with the property that x is closer to the centre of B than to any other centre in $\mathcal{B}$. It is well known that Voronoi cells are convex polyhedra, and we may in fact assume that they are polytopes. The Dodecahedral Conjecture, formulated by L. Fejes Tóth [FT43] in 1943, states that the minimal volume of a Voronoi cell in a 3-dimensional unit ball packing is at least as large as the volume of a regular dodecahedron of inradius 1. This problem has been recently settled in the affirmative by Hales and McLaughlin [HM]. K. Bezdek [Bez00] phrased the following generalised version.

Conjecture (Strong Dodecahedral Conjecture). *The minimum surface area of a Voronoi cell in a unit ball packing in $\mathbb{R}^3$ is at least as large as the surface area of the regular dodecahedron circumscribed about the unit ball, that is* $16.6508\ldots$.

In Chapter 2, we prove the following statement [AF06].

Theorem 3. *The surface area of a Voronoi cell in a unit ball packing in $\mathbb{R}^3$ is at least* $16.1977\ldots$.

This is currently the best estimate related to the problem. Prior to our result, the strongest bound was given by K. Bezdek and E. Daróczy-Kiss [BDK05], who, based on Muder's ideas ([Mud88] and [Mud93]), established the lower bound $16.1445\ldots$. Our improvement follows these lines as well.

In the proof, the cones suspended by the faces of the Voronoi cell are replaced with cones of special types in such a way, that the surface to solid angle ratio does not increase. The obtained configurations belong to a restricted class, in which the minimiser of the surface area is found by standard analytic methods.

In Section 2.2, the replacement steps are established. The cones used for replacements are the following. A *right circular cone* (RCC) is a cone whose base is a circular disk and its apex lies on the line perpendicular to the disk passing through its center. A *shaved circle* is the intersection of a disk and a convex polygon that contains the center of the disk. A *shaved right circular cone* (SRCC) is a cone whose base is a shaved circle and its apex lies on the line perpendicular to the disk and passing through its center. The desired replacements with RCC's or SRCC's are achieved via a series of basic replacement steps. Then, in Section 2.3, the surface to solid angle ratio of these special cones are further approximated.

Finally, in Section 2.4, the optimal configuration is determined using the previous approximations by a quite strenuous calculation. The minimal configuration has 13 identical faces and one face of a smaller solid angle. However, these faces cannot be joined to form a polytope, which accounts for the error between our estimate and the conjectured extremal value.

4. Stability results for the volume of random simplices

The following question serves as the motivation for Chapter 3. Given a convex body K in $\mathbb{R}^d$, what is the expected value of the volume of a random simplex in K? We work with two (or, rather, three) models: in the first, all the vertices of the simplex are chosen uniformly and independently from K, while in the second, one vertex is at a fixed position – in a special case, this is $\gamma(K)$, the centroid of K. We are interested in other moments as well, and also, we would like the answer to be invariant under affine transformations.

Definition. *Let K be a convex body in $\mathbb{R}^d$. For any $n \geqslant d+1$ and $p > 0$, let*

$$\mathbb{E}_n^p(K) = V(K)^{-n-p} \int_K \dots \int_K V([x_1, \dots, x_n])^p \, dx_1 \dots dx_n.$$

Further, for a fixed $x \in \mathbb{R}^d$, let

$$\mathbb{E}_x^p(K) = V(K)^{-d-p} \int_K \dots \int_K V([x, x_1, \dots, x_d])^p \, dx_1 \dots dx_d.$$

Specifically, we write $\mathbb{E}_*^p(K)$ for $\mathbb{E}_x^p(K)$, when $x = \gamma(K)$.

These quantities have many connections to other concepts; for example, Sylvester's problem, the volume of centroid bodies and intersection bodies, the volume of Legendre's ellipsoid, Busemann's random simplex inequality, the Busemann-Petty centroid inequality, and so on. These links are elucidated in Section 3.1.

One is mostly interested in the the minimisers and maximisers of the above expectations among convex bodies. The search of these dates back to the early 20th century, see Blaschke ([Bla17] and [Bla23]). The minimisers are known in full generality.

Theorem 4. (Blaschke [Bla23], Busemann [Bus53], Groemer[Gro74]) *For any convex body K in $\mathbb{R}^d$, for any $p \geqslant 1$, and for any $n \geqslant d+1$, we have*

$$\mathbb{E}_o^p(K) \geqslant \mathbb{E}_o^p(B^d) \quad \text{and} \quad \mathbb{E}_*^p(K) \geqslant \mathbb{E}_*^p(B^d) \quad \text{and} \quad \mathbb{E}_n^p(K) \geqslant \mathbb{E}_n^p(B^d).$$

Here $\mathbb{E}_o^p(K) = \mathbb{E}_o^p(B^d)$ if and only if K is an o-symmetric ellipsoid, *and* $\mathbb{E}_*^p(K) = \mathbb{E}_*^p(B^d)$ *or* $\mathbb{E}_n^p(K) = \mathbb{E}_n^p(B^d)$ *if and only if* K *is an ellipsoid.*

As for the maximisers, the Simplex conjecture states that for any convex body K in $\mathbb{R}^d$, and for any $p \geqslant 1$ and $n \geqslant d+1$, $\mathbb{E}_*^p(K) \leqslant \mathbb{E}_*^p(T^d)$ and $\mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(T^d)$, with equality if and only if K is a simplex. This is verified only in the plane.

THEOREM 5. ([Bla17],[DL91],[Gia92],[CCG99]) *If* $K \subset \mathbb{R}^2$ *is a convex disc, then for any* $n \geqslant 3$ *and* $p \geqslant 1$, $\mathbb{E}_n^p(K) \leqslant \mathbb{E}_n^p(T^2)$ *and* $\mathbb{E}_*^p(K) \leqslant \mathbb{E}_*^p(T^2)$, *with equality if and only if* K *is a triangle.*

The importance of the Simplex conjecture stems from the fact that the affirmative answer to it would imply the Slicing conjecture.

In Chapter 3 of the dissertation, based on [AB], we provide the corresponding stability estimates for Theorems 4 and 5. The results are formulated with the use of the *Banach-Mazur distance* $\delta_{\mathrm{BM}}(K, M)$ of the convex bodies K and M, which is defined by $\delta_{\mathrm{BM}}(K, M) = \min\{\lambda \geqslant 1 : K - x \subset \Phi(M - y) \subset \lambda(K - x)\}$, where $\Phi \in \mathrm{GL}_d$ and $x, y \in \mathbb{R}^d$. Our results are as follows.

THEOREM 6. *If* K *is a convex body in* $\mathbb{R}^d$ *with* $\delta_{\mathrm{BM}}(K, B^d) = 1 + \delta$ *for* $\delta > 0$, *then for any* $p \geqslant 1$,

$$\mathbb{E}_*^p(K) \geqslant (1 + \gamma^p \delta^{d+3}) \mathbb{E}_o^p(B^d)$$
$$\mathbb{E}_{d+1}^p(K) \geqslant (1 + \gamma^p \delta^{d+3}) \mathbb{E}_{d+1}^p(B^d),$$

where the constant $\gamma > 0$ *depends on* d *only. Moreover, if* K *is centrally symmetric, then the error terms can be replaced by* $\gamma^p \delta^{(d+3)/2}$.

Theorem 7. *If K is a planar convex body with $\delta_{\mathrm{BM}}(K, T^2) = 1 + \delta$ for some $\delta > 0$, and $p \geqslant 1$, then*

$$\begin{aligned} \mathbb{E}_*^p(K) &\leqslant (1 - c^p\delta^2)\mathbb{E}_*^p(T^2) \\ \mathbb{E}_3^p(K) &\leqslant (1 - c^p\delta^2)\mathbb{E}_3^p(T^2), \end{aligned}$$

where c is a positive absolute constant. This estimate is asymptotically sharp as δ tends to zero.

For the proof of Theorem 6, we first assume that K is a symmetric convex body in John's position, i.e. the unique ellipsoid of maximal volume inscribed in K is the unit ball. The core lemma estimates the change of the expectation when applying one step of Steiner symmetrisation in a suitable changed direction. The general result is then obtained by invoking a recent result of Böröczky [Bör], which estimates the Banach-Mazur distance between a convex body K and a symmetric convex body which is obtained by the limit of Steiner symmetrisations from K. We note that the bound of Theorem 6 is almost asymptotically sharp in terms of δ: there is an example, where the error is of order $\varepsilon^{(d+1)/2}$.

The stability version of the maximum inequality, Theorem 7 in the plane is obtained by the method of linear shadow systems, that were introduced by Campi, Colesanti and Gronchi [CCG99]. We assume that the triangle inscribed in K of maximal area is an equilateral triangle. With the aid of basic linear shadow systems, first we reduce the problem to polygons with at most 6 vertices. The main difficulty lies in the further modification of these polygons. The desired inequality is then obtained by a technical argument.

To conclude the chapter, in Section 3.6 we derive the stability version of the Petty projection inequality from Theorem 6.

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